11

Magnetic fields in matter

Overview Magnetic fields in matter are a bit more involved than electric fields in matter. Our main goal in this chapter is to understand the three types of magnetic materials: *diamagnetic* materials, which are weakly repelled by a solenoid; *paramagnetic* materials, which are somewhat strongly attracted; and *ferromagnetic* materials, which are very strongly attracted. As was the case in Chapter 10, we will need to understand dipoles. The far field of a *magnetic dipole* has the same form as that of an electric dipole, with the *magnetic dipole moment* replacing the electric dipole moment. However, the near fields are fundamentally different due to the absence of magnetic charge. We will find that diamagnetism is due to the fact that an applied magnetic field causes the magnetic dipole moment arising from the *orbital* motion of electrons in atoms to pick up a contribution pointing *opposite* to the applied field. In contrast, in the case of paramagnetism, the *spin* dipole moment is the relevant one, and it picks up a contribution pointing in the *same* direction as the applied field. Ferromagnetism is similar to paramagnetism, although a certain quantum phenomenon makes the overall effect much larger; a ferromagnetic dipole moment can exist in the absence of an external magnetic field. Magnetized materials can be described by the *magnetization* M, the curl of which gives the *bound* currents (which arise from both orbital motion and spin). By considering separately the free and bound currents, we are led to the field H (also called the "magnetic field") whose curl involves only the free current (unlike the magnetic field B, whose curl involves *all* the current, by Ampère's law).

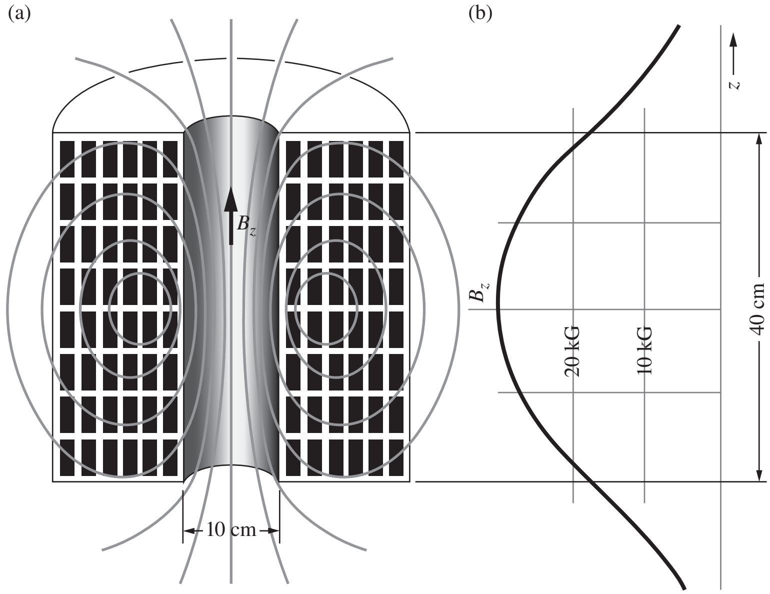


Figure 11.1.
(a) A coil designed to produce a strong magnetic field. The water-cooled winding is shown in cross section. (b) A graph of the field strength B_z on the axis of the coil.

11.1 How various substances respond to a magnetic field

Imagine doing some experiments with a very intense magnetic field. To be definite, suppose we have built a solenoid of 10 cm inside diameter, 40 cm long, like the one shown in Fig. 11.1. Its outer diameter is 40 cm, most of the space being filled with copper windings. This coil will provide a steady field of 3.0 tesla, or 30,000 gauss, at its center if supplied with 400 kilowatts of electric power – and something like 30 gallons of water per minute, to carry off the heat. We mention these practical details to show that our device, though nothing extraordinary, is a pretty respectable laboratory magnet. The field strength at the center is nearly 10^5 times the earth's field, and probably 5 or 10 times stronger than the field near any iron bar magnet or horseshoe magnet you may have experimented with, although some rare-earth magnets can have fields of around 1 tesla.

The field will be fairly uniform near the center of the solenoid, falling, on the axis at either end, to roughly half its central value. It will be rather less uniform than the field of the solenoid in Fig. 6.18, since our coil is equivalent to a "nested" superposition of solenoids with length–diameter ratio varying from 4:1 to 1:1. In fact, if we analyze our coil in that way and use the formula that we derived for the field on the axis of a solenoid with a single-layer winding (see Eq. (6.56)), it is not hard to calculate the axial field exactly. A graph of the field strength on the axis, with the central field taken as 3.0 tesla = 30 kilogauss, is included in Fig. 11.1. The intensity just at the end of the coil is 1.8 tesla, and in that neighborhood the field is changing with a gradient of approximately 17 tesla/m, or 1700 gauss/cm.

Let's put various substances into this field and see if a force acts on them. Generally, we do detect a force. It vanishes when the current in the coil is switched off. We soon discover that the force is strongest not when our sample of substance is at the center of the coil where the magnetic field B_z is strongest, but when it is located near the end of the coil where the gradient dB_z/dz is large. From now on let us support each sample just inside the upper end of the coil. Figure 11.2 shows one such sample, contained in a test tube suspended by a spring which can be calibrated to indicate the extra force caused by the magnetic field. Naturally we have to do a "blank" experiment with the test tube and suspension alone, to allow for the magnetic force on everything other than the sample.

We find in such an experiment that the force on a particular substance – metallic aluminum, for instance – is proportional to the mass of the sample and independent of its shape, as long as the sample is not too large. (Experiments with a small sample in this coil show that the force remains practically constant over a region a few centimeters in extent, inside the end of the coil; if we use samples no more than 1 to 2 cm^3 in volume, they can be kept well within this region.) We can express our quantitative results, for a given substance, as so many newtons force per kilogram of sample, under the conditions $B_z = 1.8$ tesla, $dB_z/dz = 17$ tesla/m.

But first the qualitative results, which are a bit bewildering. For a large number of quite ordinary pure substances, the force observed, although easily measurable, seems ridiculously small, despite all our effort to provide an intense magnetic field. Typically, the force is 0.1 or 0.2 newtons per kilogram, that is, no more than a few percent of the weight of the sample (which is 9.8 newtons per kilogram). It is directed upward for some samples, downward for others. This has nothing to do with the *direction* of the magnetic field, as we can verify by reversing the current in the coil. Instead, it appears that some substances are always pulled in the direction of *increasing* field intensity, others in the direction of *decreasing* field intensity, irrespective of the field direction.

We do find some substances that are attracted to the coil with considerably greater force. For instance, copper chloride crystals are pulled downward with a force of 2.8 newtons per kilogram of sample. Liquid oxygen behaves spectacularly in this experiment; it is pulled into the coil with a force nearly eight times its weight. In fact, if we were to bring an uncovered flask of liquid oxygen up to the bottom end of our coil, the liquid would be lifted right out of the flask. (Where do you think it would end up?) Liquid nitrogen, on the other hand, proves to be quite unexciting; it is pushed away from the coil with the feeble force of 0.1 newtons per kilogram. In Table 11.1 we have listed some results that one might obtain in such an experiment. The substances, including those

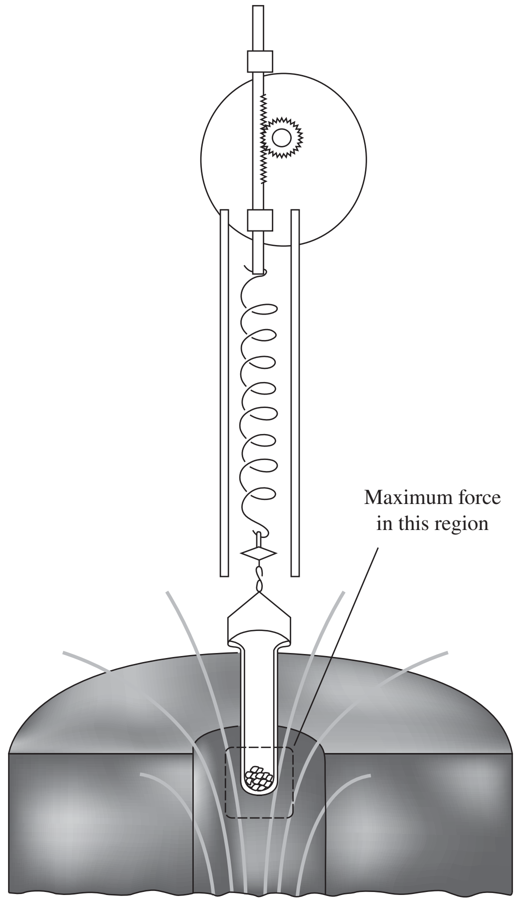


Figure 11.2.
An arrangement for measuring the force on a substance in a magnetic field.

Table 11.1.
Force per kilogram near the upper end of the coil in our experiment, where $B_z = 1.8$ tesla and $dB_z/dz = 17$ tesla/m

Substance	Formula	Force (newtons)
Diamagnetic		
Water	H_2O	−0.22
Copper	Cu	−0.026
Sodium chloride	NaCl	−0.15
Sulfur	S	−0.16
Diamond	C	−0.16
Graphite	C	−1.10
Liquid nitrogen	N_2	−0.10 (78 K)
Paramagnetic		
Sodium	Na	0.20
Aluminum	Al	0.17
Copper chloride	$CuCl_2$	2.8
Nickel sulfate	$NiSO_4$	8.3
Liquid oxygen	O_2	75 (90 K)
Ferromagnetic		
Iron	Fe	4000
Magnetite	Fe_3O_4	1200

Direction of force: downward (into coil), +; upward, −.
All measurements were made at a temperature of 20°C unless otherwise stated.
The three types of magnetism are defined in the text.

already mentioned, have been chosen to suggest, as best one can with a sparse sampling, the wide range of magnetic behavior we find in ordinary materials. Note that our convention for the sign of the force is that a positive force is directed into the coil.

As you know, a few substances, of which the most familiar is metallic iron, seem far more "magnetic" than any others. In Table 11.1 we give the force per kilogram that would act on a piece of iron put in the same position in the field as the other samples. Since 1 newton is about 0.22 pounds, the force per kilogram is roughly 900 pounds, or nearly 1 pound for a 1 gram sample! (We would not have been so naive as to approach our magnet with a gram of iron suspended in a test tube from a delicate spring – a different suspension would have to be used.) Observe that there is a factor of more than 10^5 between the force per kilogram on iron and the force per kilogram on copper, elements not otherwise radically different. Incidentally, this suggests that reliable magnetic measurements on a substance like copper may not be easy. A few parts per million contamination by metallic iron particles would utterly falsify the result.

There is another essential difference between the behavior of the iron and the magnetite and that of the other substances in the table. Suppose we make the obvious test, by varying the field strength of the magnet, to ascertain whether the force on a sample is proportional to the field. For instance, we might reduce the solenoid current by half, thereby halving both the field intensity B_z and its gradient dB_z/dz. We would find, in the case of every substance above iron in the table, that the force is reduced to *one-fourth* its former value, whereas the force on the iron sample, and that on the magnetite, would be reduced only to one-half or perhaps a little less. Evidently the force, under these conditions at least, is proportional to the square of the field strength for all the other substances listed, but nearly proportional to the field strength itself for Fe and Fe_3O_4.

It appears that we may be dealing with several different phenomena here, and complicated ones at that. As a small step toward understanding, we can introduce some classification.

(1) Diamagnetism First, those substances that are feebly *repelled* by our magnet – water, sodium chloride, diamond, etc. – are called *diamagnetic*. The majority of inorganic compounds and practically all organic compounds are diamagnetic. It turns out, in fact, that diamagnetism is a property of *every* atom and molecule. When the opposite behavior is observed, it is because the diamagnetism is outweighed by a different and stronger effect, one that leads to attraction.

(2) Paramagnetism Substances that are *attracted* toward the region of stronger field are called *paramagnetic*. In some cases, notably metals such as aluminum, sodium, and many others, the paramagnetism is not much stronger than the common diamagnetism. In other materials, such as the $NiSO_4$ and the $CuCl_2$ on our list, the paramagnetic effect is much stronger. In these substances also, it *increases* as the temperature is lowered, leading to quite large effects at temperatures near absolute zero. The increase of paramagnetism with lowering temperature is responsible in part for the large force recorded for liquid oxygen. If you think all this is going to be easy to explain, observe that copper is diamagnetic while copper chloride is paramagnetic, but sodium is paramagnetic while sodium chloride is diamagnetic.

(3) Ferromagnetism Finally, substances that behave like iron and magnetite are called *ferromagnetic*. In addition to the common metals of this class – iron, cobalt, and nickel – quite a number of ferromagnetic alloys and crystalline compounds are known. Indeed current research in ferromagnetism is steadily lengthening the list.

In this chapter we have two tasks. One is to develop a treatment of the large-scale phenomena involving magnetized matter, in which the material itself is characterized by a few parameters and the experimentally determined relations among them. It is like a treatment of dielectrics based on some observed relation between electric field and bulk polarization. We sometimes call such a theory *phenomenological;* it is more of a description than an explanation. Our second task is to try to understand, at least in a general way, the atomic origin of the various magnetic effects. Even more than dielectric phenomena, the magnetic effects, once understood, reveal some basic features of atomic structure.

One general fact stands out in Table 11.1. Very little energy, on the scale of molecular energies, is involved in diamagnetism and paramagnetism. Take the extreme example of liquid oxygen. To pull 1 kilogram (although the sample size would certainly be much smaller) of liquid oxygen away from our magnet, energy would have to be expended amounting, in joules, to 75 newtons times a distance of roughly 0.1 meters (since the field strength falls off substantially over a distance of a few centimeters). In order of magnitude, let us say the energy is 10 joules. There are $2 \cdot 10^{25}$ molecules in 1 kilogram of the liquid, so this is less than 10^{-24} joules per molecule. Just to vaporize 1 kilogram of liquid oxygen requires 50,000 calories, or about 10^{-20} joules per molecule, using 1 calorie = 4.18 joules. (Most of that energy is used in separating the molecules from one another.) Whatever may be happening in liquid oxygen at the molecular level as a result of the magnetic field, it is apparently a very minor affair in terms of energy.

Even a strong magnetic field has hardly any effect on chemical processes, including biochemical. You could put your hand and forearm into our 3 tesla solenoid without experiencing any significant sensation or consequence. It is hard to predict whether your arm would prove to be paramagnetic or diamagnetic, but the force on it would be no more than a fraction of an ounce in any case. Conversely, the presence of someone's hand close to the sample in Fig. 11.2 would perturb the field and change the force on the sample by no more than a few parts in a million. In whole-body imaging with nuclear magnetic resonance, the body is pervaded by a magnetic field of a few tesla in strength with no physiological effects whatsoever. It appears that the only hazards associated with large-scale, strong, steady magnetic fields arise from metal objects in the vicinity. For example, implants containing metal may heat up, move within the body, or malfunction. And there is also the danger that a loose iron object will be snatched by the fringing field and hurled into the magnet. Be careful what you bring into a magnetic resonance imaging (MRI) room!

In its interaction with matter, the magnetic field plays a role utterly different from that of the electric field. The reason is simple and fundamental. Atoms and molecules are made of electrically charged particles that move with velocities generally small compared with the speed of light. A magnetic field exerts no force at all on a stationary electric charge; on a moving charged particle the force is proportional to v^2/c^2.[1]

[1] This factor of v^2/c^2 follows from Eq. (5.28). The current I in that equation involves the velocity of charges, and we are assuming that all velocities here are of the same order of magnitude. Of course, if we have a charged particle moving in a region where the electric field is zero and the magnetic field is nonzero, then the magnetic field dominates. But for general random motions of the charges (both the charges creating the fields and the charges affected by the fields), the magnetic force is smaller by a factor of v^2/c^2.

Said in a sloppier way, in SI units the $1/4\pi\epsilon_0$ factor in Coulomb's law for the electric field is large, while the $\mu_0/4\pi$ factor in the Biot–Savart law for the magnetic field is small. Electric forces overwhelmingly dominate the atomic scene. As we have remarked before, magnetism appears, in our world at least, to be a relativistic effect. The story would be dif-ferent if matter were made of magnetically charged particles. We must explain now what *magnetic charge* means and what its apparent absence signifies.

11.4 The force on a dipole in an external field

Consider a small circular current loop of radius r, placed in the magnetic field of some other current system, such as a solenoid. In Fig. 11.9, a field

B is drawn that is generally in the z direction. It is not a uniform field. Instead, it gets weaker as we proceed in the z direction; this is evident from the fanning out of the field lines. Let us assume, for simplicity, that the field is symmetric about the z axis. Then it resembles the field near the upper end of the solenoid in Fig. 11.1. The field represented in Fig. 11.9 does *not* include the magnetic field of the current ring itself. We want to find the force on the current ring caused by the other field, which we shall call, for want of a better name, the *external field.* The net force on the current ring due to its *own* field is certainly zero, so we are free to ignore its own field in this discussion.

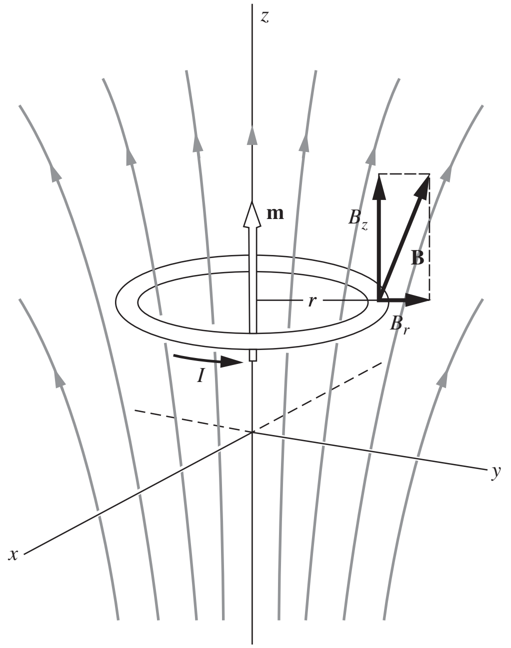


Figure 11.9.
A current ring in an inhomogeneous magnetic field. (The field of the ring itself is not shown.) Because of the radial component of the field, B_r, there is a force on the ring as a whole.

If you study the situation in Fig. 11.9, you will soon conclude that there is a net force on the current ring. It arises because the external field **B** has an outward component B_r everywhere around the ring. (The vertical component B_z produces a force in the horizontal plane that simply stretches or compresses the ring – negligibly, assuming the ring is fairly rigid.) Therefore if the current flows in the direction indicated, each element of the loop, dl, must be experiencing a downward force of magnitude $IB_r\,dl$ (see Eq. (6.14)). If B_r has the same magnitude at all points on the ring, as it must in the symmetrically spreading field assumed, the total downward force will have the magnitude

$$F = 2\pi r I B_r. \tag{11.16}$$

Now, B_r can be directly related to the gradient of B_z. Since $\text{div}\,\mathbf{B} = 0$ at all points, the net flux of magnetic field out of any volume is zero. Consider a pancake-like cylinder of radius r and height Δz (Fig. 11.10). The outward flux from the side is $2\pi r(\Delta z)B_r$ and the net outward flux from the end surfaces is

$$\pi r^2[-B_z(z) + B_z(z+\Delta z)], \tag{11.17}$$

which to the first order in the small distance Δz is $\pi r^2(\partial B_z/\partial z)\Delta z$. Setting the total outward flux equal to zero gives $0 = \pi r^2(\partial B_z/\partial z)\Delta z + 2\pi r B_r \Delta z$, or

$$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}. \tag{11.18}$$

As a check on the sign, note that, according to Eq. (11.18), B_r is positive when B_z is decreasing upward; a glance at the figure shows that to be correct.

The force on the dipole (with upward taken to be positive) can now be expressed in terms of the gradient of the component B_z of the external field:

$$F = -2\pi r I\left(-\frac{r}{2}\frac{\partial B_z}{\partial z}\right) = \pi r^2 I \frac{\partial B_z}{\partial z}. \tag{11.19}$$

In the present case, $\partial B_z/\partial z$ is negative, so the force is correctly downward. In the factor $\pi r^2 I$ we recognize the magnitude m of the magnetic

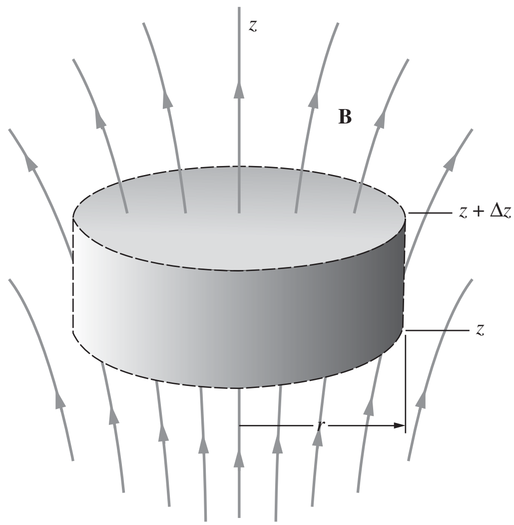


Figure 11.10.
Gauss's theorem can be used to relate B_r and $\partial B_z/\partial z$, leading to Eq. (11.18).

dipole moment of our current ring. So the force on the ring can be expressed very simply in terms of the dipole moment:

$$\boxed{F = m\frac{\partial B_z}{\partial z}} \tag{11.20}$$

We haven't proved it, but you will not be surprised to hear that for small loops of any other shape the force depends only on the *current* × *area* product, that is, on the dipole moment. The shape doesn't matter. Of course, we are discussing only loops small enough so that only the first-order variation of the external field, over the span of the loop, is significant.

Our ring in Fig. 11.9 has a magnetic dipole moment **m** pointing upward, and the force on it is downward. Obviously, if we could reverse the current in the ring, thereby reversing **m**, the force would reverse its direction. The situation can be summarized as follows.

- Dipole moment *parallel* to external field: force acts in direction of *increasing* field strength.
- Dipole moment *antiparallel* to external field: force acts in direction of *decreasing* field strength.
- *Uniform* external field: *zero* force.

Quite obviously, this is not the most general situation. The moment **m** could be pointing at some odd angle with respect to the field **B**, and the different components of **B** could be varying, spatially, in different ways. Given all the similarities between electric and magnetic dipoles, it is tempting to say that the force on a magnetic dipole should take the same form as the force on an electric dipole, given in Eq. (10.26). That is, the x component of the force on a magnetic dipole **m** should be given by

$$F_x = \mathbf{m} \cdot \nabla B_x \qquad \text{(incorrect)}, \tag{11.21}$$

with corresponding formulas for F_y and F_z. All three components can be combined into the compact relation,

$$\mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B} \qquad \text{(incorrect)}. \tag{11.22}$$

You can check in Problem 11.4 that in the above setup with the ring, this force reduces to the force in Eq. (11.20).

However, this argument by analogy is risky, because, although the fields due to electric and magnetic dipoles look the same at large distances, the dipoles themselves look very different up close. One consists of two point charges, the other of a loop of current. The far field is irrelevant when dealing with the force *on* a dipole. It turns out that, although Eq. (11.22) gives the correct force on a magnetic dipole in many cases, it is *not* correct in general. The correct expression for the force turns out to be

$$\boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})} \tag{11.23}$$

You can check in Problem 11.4 that in the above setup, this also reduces to the force in Eq. (11.20). At first glance it might seem like Eq. (11.23) comes out of the blue, but there is actually very good motivation for it. We will see in Section 11.6 that the energy of a magnetic dipole in a magnetic field is $-\mathbf{m} \cdot \mathbf{B}$. (But see Feynman *et al.* (1977), chap. 15, for a discussion of a subtlety about this energy.) So Eq. (11.23) is the familiar statement that the force equals the negative gradient of the energy.

Under what conditions are the force expressions in Eqs. (11.22) and (11.23) equal? Using the "$\nabla(\mathbf{A}\cdot\mathbf{B})$" vector identity in Appendix K, along with the fact that $\mathbf{m}$ has no spatial dependence, we find

$$\nabla(\mathbf{m} \cdot \mathbf{B}) = (\mathbf{m} \cdot \nabla)\mathbf{B} + \mathbf{m} \times (\nabla \times \mathbf{B}). \qquad (11.24)$$

Our two expressions for the force are therefore equal if $\nabla \times \mathbf{B} = 0$.[3] If we deal only with static setups where $\partial \mathbf{E}/\partial t = 0$, then the relevant Maxwell equation reduces to Ampère's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. So we see that the two expressions for the force agree if the setup involves no currents at the location of the dipole (other than the current in the dipole loop itself). This was the case in the above example. However, Problem 11.4 presents a setup where Eqs. (11.22) and (11.23) yield different forces; the task of that problem is to calculate the force explicitly and show that it agrees with Eq. (11.23).

In Eqs. (11.20) and (11.23) the force is in newtons, with the magnetic field gradient in tesla/meter and the magnetic dipole moment m given by Eq. (11.9): $m = Ia$, where I is in amps and a in m^2. There are several equivalent ways to express the units of m. From Eq. (11.9) the units are

$$[m] = \text{amp-m}^2. \qquad (11.25)$$

But, as you can see from Eq. (11.20), we also have

$$[m] = \frac{\text{newtons}}{\text{tesla/m}} = \frac{\text{newton-m}}{\text{tesla}} = \frac{\text{joules}}{\text{tesla}}. \qquad (11.26)$$

Looking back at the three cases summarized on p. 538, we can begin to see what must be happening in the experiments described at the beginning of this chapter. A substance located at the position of the sample in Fig. 11.2 would be attracted *into* the solenoid if it contained magnetic dipoles *parallel* to the field $\mathbf{B}$ of the coil. It would be pushed *out* of the solenoid if it contained dipoles pointing in the opposite direction, *antiparallel* to the field. The force would depend on the gradient of the axial field strength, and would be zero at the midpoint of the solenoid. Also, if the total strength of dipole moments in the sample were proportional to the field strength $\mathbf{B}$, then in a given position the force would be proportional to B times $\partial B/\partial z$, and hence to the square of the solenoid current. This is the observed behavior in the case of the diamagnetic

[3] This is a sufficient condition, but not necessary. Technically all we need is for $\nabla \times \mathbf{B}$ to be parallel to $\mathbf{m}$. But if we want the two expressions to be equal for any orientation of $\mathbf{m}$, then we need $\nabla \times \mathbf{B} = 0$.

and the paramagnetic substances. It looks as if the ferromagnetic samples must have possessed a magnetic moment nearly independent of field strength, but we must set them aside for a special discussion anyway.

How does the application of a magnetic field to a substance evoke in the substance magnetic dipole moments with total strength proportional to the applied field? And why should they be parallel to the field in some substances, and oppositely directed in others? If we can answer these questions, we shall be on the way to understanding the physics of diamagnetism and paramagnetism.

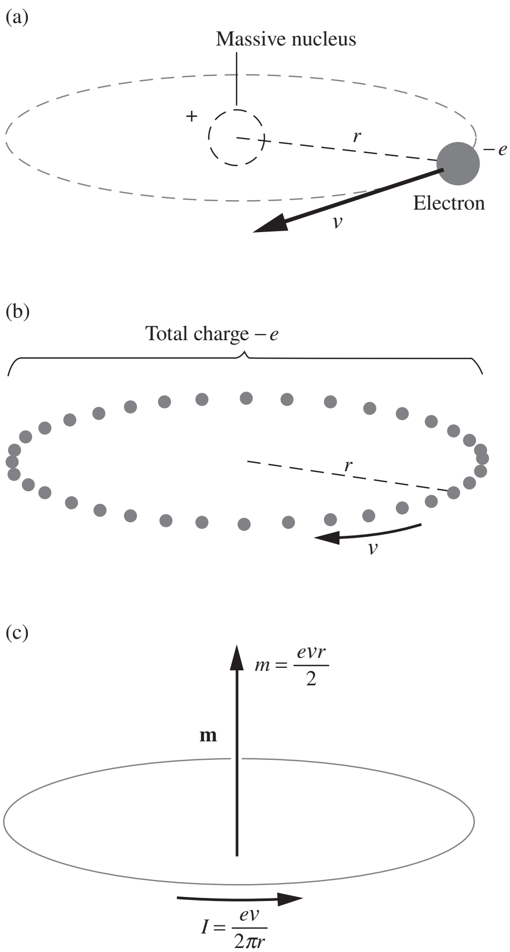


Figure 11.11.
(a) A model of an atom in which one electron moves at speed v in a circular orbit.
(b) Equivalent procession of charge. The average electric current is the same as if the charge $-e$ were divided into small bits, forming a rotating ring of charge. (c) The magnetic moment is the product of current and area.

11.5 Electric currents in atoms

We know that an atom consists of a positive nucleus surrounded by negative electrons. To describe it fully we would need the concepts of quantum physics. Fortunately, a simple and easily visualized model of an atom is very helpful for understanding diamagnetism. It is a planetary model with the electrons in orbits around the nucleus, like the model in Bohr's first quantum theory of the hydrogen atom.

We begin with one electron moving at constant speed on a circular path. Since we are not attempting here to explain atomic structure, we shall not inquire into the reasons why the electron has this particular orbit. We ask only, if it does move in such an orbit, what magnetic effects are to be expected? In Fig. 11.11 we see the electron, visualized as a particle carrying a concentrated electric charge $-e$, moving with speed v on a circular path of radius r. In the middle is a positive nuclear charge, making the system electrically neutral. But the nucleus, because of its relatively great mass, moves so slowly that its magnetic effects can be neglected.

At any instant, the electron and the positive charge would appear as an electric dipole, but on the time average the electric dipole moment is zero, producing no steady electric field at a distance. We discussed this point in Section 10.5. The *magnetic* field of the system, far away, is *not* zero on the time average. Instead, it is just the field of a current ring. This is because, when considering the time average, it can't make any difference whether we have all the negative charge gathered into one lump, going around the track, or distributed in bits, as in Fig. 11.11(b), to make a uniform endless procession. The current is the amount of charge that passes a given point on the ring, per second. Since the electron makes $v/2\pi r$ revolutions per second, the current is

$$I = \frac{ev}{2\pi r}. \tag{11.27}$$

The orbiting electron is equivalent to a ring current of this magnitude with the direction of positive flow opposite to v, as shown in Fig. 11.11(c). Its far field is therefore that of a magnetic dipole, of strength

$$m = \pi r^2 I = \frac{evr}{2}. \tag{11.28}$$

Let us note in passing a simple relation between the magnetic moment **m** associated with the electron orbit, and the orbital angular momentum **L**. The angular momentum is a vector of magnitude $L = m_e vr$, where m_e denotes the mass of the electron,[4] and it points downward if the electron is revolving in the sense shown in Fig. 11.11(a). Note that the product vr occurs in both m and L. With due regard to direction, we can write

$$\boxed{\mathbf{m} = \frac{-e}{2m_e}\mathbf{L}} \tag{11.29}$$

This relation involves nothing but fundamental constants, which should make you suspect that it holds quite generally. Indeed that is the case, although we shall not prove it here. It holds for elliptical orbits, and it holds even for the rosette-like orbits that occur in a central field that is not inverse-square. Remember the important property of any orbit in a central field: angular momentum is a constant of the motion. It follows then, from the general relation expressed by Eq. (11.29) (derived by us only for a special case), that wherever angular momentum is conserved, the magnetic moment also remains constant in magnitude and direction. The factor

$$\frac{-e}{2m_e} \quad \text{or} \quad \frac{\text{magnetic moment}}{\text{angular momentum}} \tag{11.30}$$

is called the *orbital magnetomechanical ratio* for the electron.[5] The intimate connection between magnetic moment and angular momentum is central to any account of atomic magnetism.

Why don't we notice the magnetic fields of all the electrons orbiting in all the atoms of every substance? The answer must be that there is a mutual cancelation. In an ordinary lump of matter there must be as many electrons going one way as the other. This is to be expected, for there is nothing to make one sense of rotation intrinsically easier than another, or otherwise to distinguish any unique axial direction. There would have to be something in the structure of the material to single out not merely an axis, but a *sense of rotation around that axis!*

We may picture a piece of matter, in the absence of any external magnetic field, as containing revolving electrons with their various orbital angular momentum vectors and associated orbital magnetic moments

[4] Our choice of the symbol **m** for magnetic moment makes it necessary, in this chapter, to use a different symbol for the electron mass. For angular momentum we choose the symbol **L**, because **L** is traditionally used in atomic physics for orbital angular momentum, which is what we consider here. We shall be dealing with speeds v much less than c, so the nonrelativistic expression for **L** will suffice.

[5] Many people use the term *gyromagnetic ratio* for this quantity. Some call it the *magnetogyric ratio*. Whatever the name, it is understood that the magnetic moment is the numerator.

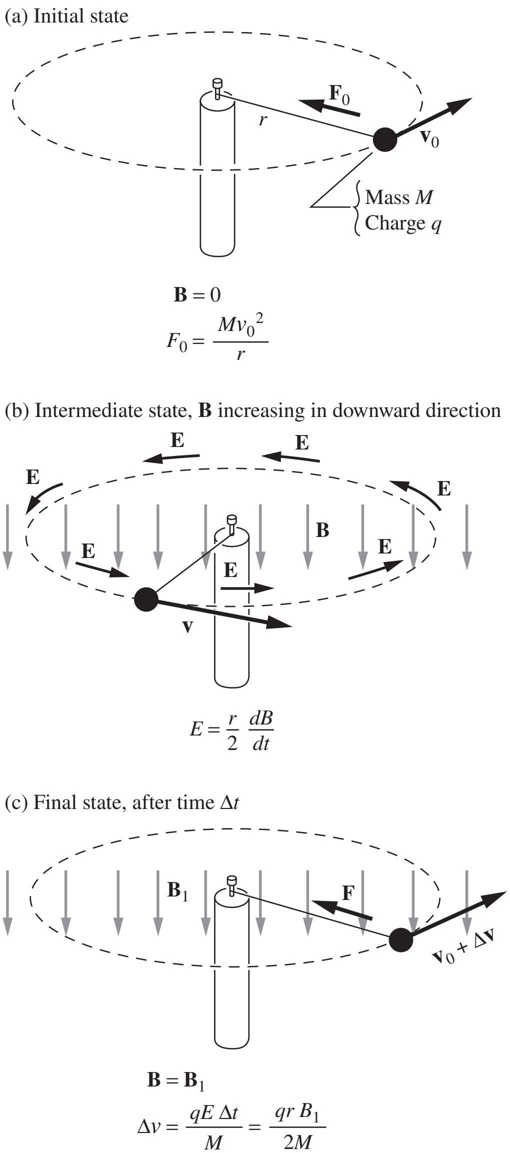


Figure 11.12.
The growth of the magnetic field **B** induces an electric field **E** that accelerates the revolving charged body.

distributed evenly over all directions in space. Consider those orbits that happen to have their planes approximately parallel to the xy plane, of which there will be about equal numbers with **m** up and **m** down. Let's find out what happens to one of these orbits when we switch on an external magnetic field in the z direction.

We will start by analyzing an electromechanical system that doesn't look much like an atom. In Fig. 11.12 there is an object of mass M and electric charge q, tethered to a fixed point by a cord of fixed length r. This cord provides the centripetal force that holds the object in its circular orbit. The magnitude of that force F_0 is given, as we know, by

$$F_0 = \frac{Mv_0^2}{r}. \tag{11.31}$$

In the initial state, Fig. 11.12(a), there is no external magnetic field. Now, by means of some suitable large solenoid, we begin creating a field **B** in the negative z direction, uniform over the whole region at any given time. While this field is growing at the rate dB/dt, there will be an induced electric field **E** all around the path, as indicated in Fig. 11.12(b). To find the magnitude of this field **E** we note that the rate of change of flux through the circular path is

$$\frac{d\Phi}{dt} = \pi r^2 \frac{dB}{dt}. \tag{11.32}$$

This determines the line integral of the electric field, which is really all that matters (we only assume for symmetry and simplicity that it is the same all around the path). Faraday's law, $\mathcal{E} = -d\Phi/dt$, gives (ignoring the signs)

$$\int \mathbf{E} \cdot d\boldsymbol{l} = \pi r^2 \frac{dB}{dt}. \tag{11.33}$$

The left-hand side equals $2\pi rE$, so we find that

$$E = \frac{r}{2}\frac{dB}{dt}. \tag{11.34}$$

We have ignored signs so far, but if you apply to Fig. 11.12 your favorite rule for finding the direction of an induced electromotive force, you will see that **E** must be in a direction to accelerate the body, if q is a positive charge. The acceleration along the path, dv/dt, is determined by the force qE:

$$M\frac{dv}{dt} = qE = \frac{qr}{2}\frac{dB}{dt}, \tag{11.35}$$

so that we have a relation between the change in v and the change in B:

$$dv = \frac{qr}{2M}\,dB. \tag{11.36}$$

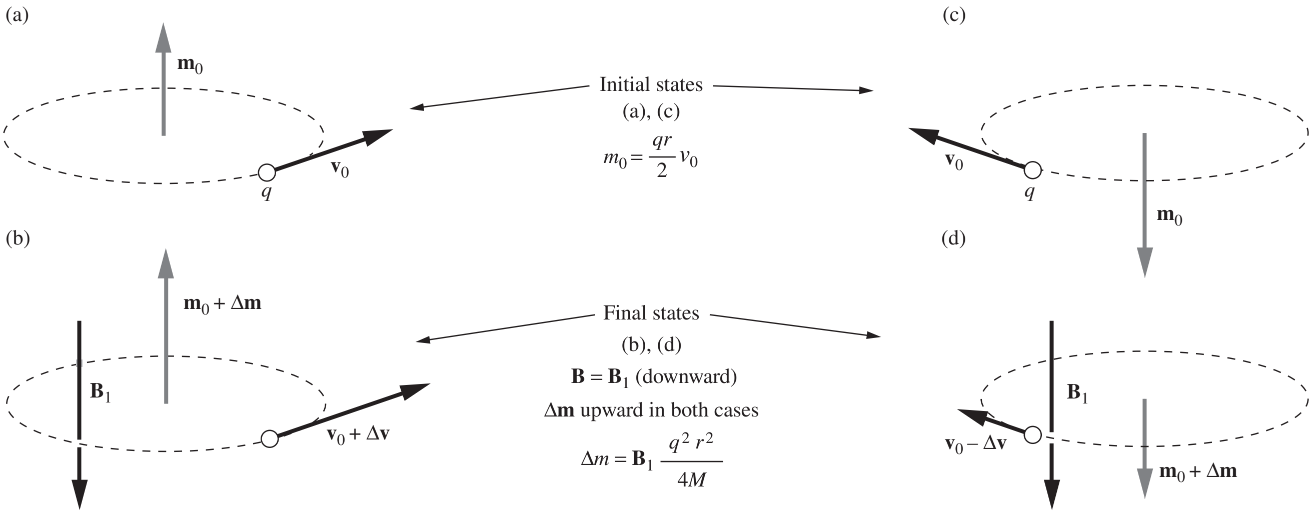


Figure 11.13.
The change in the magnetic moment vector is opposite to the direction of **B**, for both directions of motion.

The radius r being fixed by the length of the cord, the factor $qr/2M$ is a constant. Let Δv denote the net change in v in the whole process of bringing the field up to the final value B_1. Then

$$\Delta v = \int_{v_0}^{v_0+\Delta v} dv = \frac{qr}{2M} \int_0^{B_1} dB = \frac{qrB_1}{2M}. \qquad (11.37)$$

Note that the time has dropped out – the final velocity is the same whether the change is made slowly or quickly.

The increased speed of the charge in the final state means an increase in the upward-directed magnetic moment **m**. A *negatively* charged body would have been *de*celerated under similar circumstances, which would have *decreased* its *downward* moment. In either case, then, the application of the field $\mathbf{B}_1$ has brought about a change in magnetic moment opposite to the field. From Eq. (11.28), the magnitude of the change in magnetic moment Δm is

$$\Delta m = \frac{qr}{2} \Delta v = \frac{q^2 r^2}{4M} B_1. \qquad (11.38)$$

Likewise for charges, either positive or negative, revolving in the other direction, the induced change in magnetic moment is opposite to the change in applied magnetic field. Figure 11.13 shows this for a positive charge. It appears that the following relation holds for either sign of charge and either direction of revolution:

$$\boxed{\Delta \mathbf{m} = -\frac{q^2 r^2}{4M} \mathbf{B}_1} \qquad (11.39)$$

In this example we forced r to be constant by using a cord of fixed length. Let us see how the tension in the cord has changed. We shall assume that B_1 is small enough so that $\Delta v \ll v_0$. In the final state we require a centripetal force of magnitude

$$F_1 = \frac{M(v_0 + \Delta v)^2}{r} \approx \frac{Mv_0^2}{r} + \frac{2Mv_0\Delta v}{r}, \tag{11.40}$$

neglecting the term proportional to $(\Delta v)^2$. But now the magnetic field itself provides an inward force on the moving charge, given by $q(v_0 + \Delta v)B_1$. Using Eq. (11.37) to express qB_1 in terms of Δv, we find that this extra inward force has the magnitude $(v_0 + \Delta v)(2M\Delta v/r)$ which, to first order in $\Delta v/v_0$, is $2Mv_0\,\Delta v/r$. That is just what is needed, according to Eq. (11.40), to avoid any extra demand on our cord! Hence the tension in the cord *remains unchanged at the value* F_0.

This points to a surprising conclusion: our result, Eq. (11.39), must be valid for *any* kind of tethering force, no matter how it varies with radius. Our cord could be replaced by an elastic spring without affecting the outcome – the radius would still be unchanged in the final state. Or to go at once to a system we are interested in, it could be replaced by the Coulomb attraction of a nucleus for an electron. Or it could be the effective force that acts on one electron in an atom containing many electrons, which has a still different dependence on radius.

Let us apply this to an electron in an atom, substituting the electron mass m_e for M, and e^2 for q^2. Now Δm is the magnetic moment induced by the application of a field B_1 to the atom. In other words, $\Delta m/B_1$ is a magnetic polarizability, defined in the same way as the electric polarizability α we introduced in Section 10.5. Remember that $\alpha/4\pi\epsilon_0$ had the dimensions of volume and turned out to be, in order of magnitude, 10^{-30} m^3, roughly the volume of an atom. By Eq. (11.39), the magnetic polarizability due to one electron in an orbit of radius r is

$$\frac{\Delta m}{B_1} = -\frac{e^2 r^2}{4m_e}. \tag{11.41}$$

Taking the orbit radius r to be the Bohr radius, $0.53 \cdot 10^{-10}$ m, we find

$$\frac{\Delta m}{B_1} = -\frac{(1.6 \cdot 10^{-19}\,\text{C})^2 (0.53 \cdot 10^{-10}\,\text{m})^2}{4(9.1 \cdot 10^{-31}\,\text{kg})} = -2 \cdot 10^{-29}\,\frac{\text{C}^2\,\text{m}^2}{\text{kg}}. \tag{11.42}$$

However, this comparison between Δm and B_1 isn't quite a fair one, because magnetic fields contain a somewhat arbitrary factor of $\mu_0/4\pi$ multiplying the factors of current and distance; see the Biot–Savart law in Eq. (6.49). (An analogous issue arose with the electric polarizability in Section 10.5.) A more reasonable comparison would therefore involve $(\mu_0/4\pi)\Delta m$ and B_1. Since $\mu_0/4\pi = 1 \cdot 10^{-7}$ kg m/C^2, the numerical value of the ratio is simply modified by a factor of 10^{-7}, and we have

$$\frac{\mu_0}{4\pi}\frac{\Delta m}{B_1} = -2 \cdot 10^{-36}\,\text{m}^3. \tag{11.43}$$

This has the dimensions of volume, just like the electric polarizability $\alpha/4\pi\epsilon_0$. However, it is five or six orders of magnitude smaller than typical electric polarizabilities, as sampled in Table 10.2. We can be a little more precise about this disparity. Using Eq. (11.41), we have (recalling $\mu_0 = 1/\epsilon_0 c^2$)

$$\frac{\mu_0}{4\pi}\frac{\Delta m}{B_1} = -\frac{r^2}{4}\frac{\mu_0 e^2}{4\pi m_e} = -\frac{r^2}{4}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{m_e c^2}\right) \equiv -\frac{r^2}{4} r_0. \qquad (11.44)$$

The quantity in parentheses has the dimensions of length and is known as the *classical electron radius*,[6] $r_0 = 2.8 \cdot 10^{-15}$ m. Since the electric polarizability is given by $\alpha/4\pi\epsilon_0 \approx r^3$, where r is an atomic radius, we see that the magnetic polarizability is smaller than the electric polarizability by the ratio (roughly, up to a factor of 4 and other factors of order 1 that our models ignore) of the classical electron radius r_0 to an atomic radius r.

Example Let us see if Eq. (11.41) will account for the force on our diamagnetic samples listed in Table 11.1. The total number of electrons is about the same in one gram of almost anything. It is about one electron for every two nucleons (recall that the atomic weight is about twice the atomic number for most of the elements), or $N = 3 \cdot 10^{26}$ electrons per kilogram of matter. This follows from the fact that the mass of a nucleon is $1.67 \cdot 10^{-27}$ kg, so there are $6 \cdot 10^{26}$ nucleons in a kilogram. Equivalently, hydrogen has an atomic weight of 1, so the number of nucleons in one gram is Avogadro's number, $6.02 \cdot 10^{23}$.

Of course, the r^2 in Eq. (11.41) must now be replaced by a mean square orbit radius $\langle r^2 \rangle$, where the average is taken over all the electrons in the atom, some of which have larger orbits than others. Actually $\langle r^2 \rangle$ varies remarkably little from atom to atom through the whole periodic table, and a_0^2, the square of the Bohr radius which we have just used, remains a surprisingly good estimate. Adopting that, we would predict, using Eq. (11.42), that a field of 1.8 tesla would induce in 1 kg of substance a magnetic moment of magnitude

$$\begin{aligned}\Delta m = N\frac{e^2 r^2}{4m_e}B_1 &= (3 \cdot 10^{26})(2 \cdot 10^{-29}\,\text{C}^2\,\text{m}^2/\text{kg})(1.8\,\text{tesla}) \\ &= 1.1 \cdot 10^{-2}\,\frac{\text{joule}}{\text{tesla}} \quad (\text{or amp-m}^2),\end{aligned} \qquad (11.45)$$

which in a gradient of 17 tesla/m would give rise to a force of magnitude

$$F = \Delta m \frac{\partial B_z}{\partial z} = \left(1.1 \cdot 10^{-2}\,\frac{\text{joule}}{\text{tesla}}\right)\left(17\,\frac{\text{tesla}}{\text{m}}\right) = 0.18\,\text{newtons}. \qquad (11.46)$$

This agrees quite well, indeed better than we had any right to expect, with the values for the several purely diamagnetic substances listed in Table 11.1. As far as the sign of the force goes, we know from Eq. (11.39) that the magnetic moment is

[6] This radius is obtained by setting the rest energy of the electron, $m_e c^2$, equal to (in order of magnitude) the electrostatic potential energy of a ball with radius r_0 and charge e. See Exercise 1.62.

antiparallel to the external magnetic field. The discussion in Section 11.4 therefore tells us that the force acts in the direction of decreasing field strength, that is, outward from the solenoid. This agrees with the behavior of the diamagnetic samples, because the convention in Table 11.1 was that "−" meant outward.

We can see now why diamagnetism is a universal phenomenon, and a rather inconspicuous one. It is about the same in molecules as in atoms. The fact that a molecule can be a much larger structure than an atom – it may be built of hundreds or thousands of atoms – does not generally increase the effective mean-square orbit radius. The reason is that in a molecule any given electron is pretty well localized on an atom. There are some interesting exceptions, and we included one in Table 11.1 – graphite. The anomalous diamagnetism of graphite is due to an unusual structure that permits some electrons to circulate rather freely within a planar group of atoms in the crystal lattice. For these electrons $\langle r^2 \rangle$ is extraordinarily large.

As mentioned at the beginning of this section, diamagnetism (and likewise paramagnetism and ferromagnetism) can be explained only with quantum mechanics. A purely classical theory of diamagnetism does not exist; see O'Dell and Zia (1986). Nevertheless, the above discussion is helpful for understanding the critical property of diamagnetism, namely that the change in the magnetic moment is directed opposite to the applied magnetic field.

11.6 Electron spin and magnetic moment

In addition to its orbital angular momentum, the electron possesses another kind of angular momentum that has nothing to do with its orbital motion. It behaves in many ways as if it were continually rotating around an axis of its own. This property is called *spin.* While diamagnetism is a result of the orbital angular momentum of electrons, paramagnetism is a result of their spin angular momentum (as is ferromagnetism, which we will discuss in Section 11.11). A consequence of these origins is that a diamagnetic moment points antiparallel to the external magnetic field, whereas a paramagnetic moment points parallel to the external field (in an average sense, as we will see).

When the magnitude of the spin angular momentum is measured, the same result is always obtained: $h/4\pi$, where h is Planck's constant, $6.626 \cdot 10^{-34}\,\text{kg m}^2/\text{s}$. Electron spin is a quantum phenomenon. Its significance for us now lies in the fact that there is associated with this intrinsic, or "built-in," angular momentum a *magnetic moment,* likewise of invariable magnitude. This magnetic moment points in the direction you would expect if you visualize the electron as a ball of negative charge spinning around its axis. That is, the magnetic moment vector points antiparallel to the spin angular momentum vector, as indicated in

Fig. 11.14. The magnetic moment, however, is twice as large, relative to the angular momentum, as it is in the case of orbital motion.

There is no point in trying to devise a classical model of this object; its properties are essentially quantum mechanical. We need not even go so far as to say it *is* a current loop. What matters is only that it behaves like one in the following respects: (1) it produces a magnetic field that, at a distance, is that of a magnetic dipole; (2) in an external field $\mathbf{B}$ it experiences a torque equal to that which would act on a current loop of equivalent dipole moment; (3) within the space occupied by the electron, $\text{div}\,\mathbf{B} = 0$ everywhere, as in the ordinary sources of magnetic field with which we are already familiar.

Since the magnitude of the spin magnetic moment is always the same, the only thing an external field can influence is its direction. (Contrast this with the changing magnitude of the orbital magnetic moment in Section 11.5.) A magnetic dipole in an external field experiences a torque. If you worked through Exercise 6.34, you proved that the torque $\mathbf{N}$ on a current loop of any shape, with dipole moment $\mathbf{m}$, in a field $\mathbf{B}$, is given by

$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B}} \tag{11.47}$$

For those who have not been through that demonstration, let's take time out to calculate the torque in a simple special case. In Fig. 11.15 we see a rectangular loop of wire carrying current I. The loop has a magnetic moment $\mathbf{m}$, of magnitude $m = Iab$. The torque on the loop arises from the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ that act on the horizontal wires. Each of these forces has the magnitude $F = IbB$, and its moment arm is the distance $(a/2)\sin\theta$. We see that the magnitude of the torque on the loop is

$$N = 2(IbB)\frac{a}{2}\sin\theta = (Iab)B\sin\theta = mB\sin\theta. \tag{11.48}$$

The torque acts in a direction to bring $\mathbf{m}$ parallel to $\mathbf{B}$; it is represented by a vector $\mathbf{N}$ in the positive x direction, in the situation shown. All this is consistent with the general formula, Eq. (11.47). Note that Eq. (11.47) corresponds exactly to the formula we derived in Section 10.4 for the torque on an electric dipole $\mathbf{p}$ in an external field $\mathbf{E}$, namely, $\mathbf{N} = \mathbf{p} \times \mathbf{E}$. The orientation with $\mathbf{m}$ in the direction of $\mathbf{B}$, like that of the electric dipole $\mathbf{p}$ parallel to $\mathbf{E}$, is the orientation of lowest energy. Similarly, the work required to rotate a dipole $\mathbf{m}$ from an orientation parallel to $\mathbf{B}$, through an angle θ_0, is $mB(1 - \cos\theta_0)$; see Eq. (10.22). This equals $2mB$ for a rotation from parallel to antiparallel.

If the electron spin moments in a substance are free to orient themselves, we expect them to prefer the orientation in the direction of any applied field $\mathbf{B}$, the orientation of lowest energy. Suppose every electron in a kilogram of material takes up this orientation. We have already calculated that there are roughly $3 \cdot 10^{26}$ electrons in a kilogram of anything. The spin magnetic moment of an electron, m_s, is given in Fig. 11.14 as

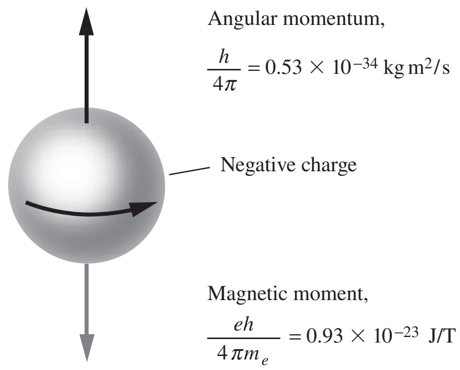


Figure 11.14.
The intrinsic angular momentum, or spin, and the associated magnetic moment, of the electron. Note that the ratio of magnetic moment to angular momentum is e/m_e, not $e/2m_e$ as it is for orbital motion; see Eq. (11.29). This has no classical explanation.

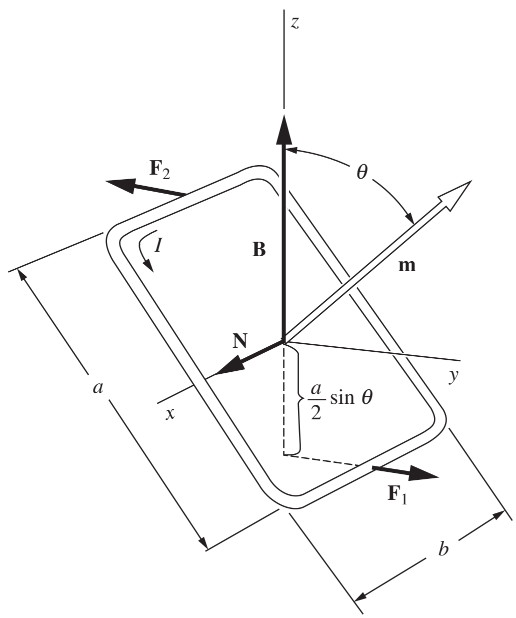


Figure 11.15.
Calculation of the torque on a current loop in a magnetic field $\mathbf{B}$. The magnetic moment of the current loop is $\mathbf{m}$.

$$m_s = 9.3 \cdot 10^{-24} \ \frac{\text{joules}}{\text{tesla}} \quad (\text{or amp-m}^2). \tag{11.49}$$

The total magnetic moment of our lined-up spins in one kilogram will be $(3 \cdot 10^{26}) \times (9.3 \cdot 10^{-24})$ or 2800 joules/tesla. From Eq. (11.20), the force per kilogram, in our coil where the field gradient is 17 tesla/m, would be $4.7 \cdot 10^4$ newtons. This is a little over 10,000 pounds, or equivalently 10 pounds for a tiny 1 gram sample!

Obviously this is much greater than the force recorded for any of the paramagnetic samples. Our assumptions were wrong in two ways. First, the electron spin moments are not all free to orient themselves. Second, thermal agitation prevents perfect alignment of any spin moments that are free. Let us look at these two issues in turn.

In most atoms and molecules, the electrons are associated in pairs, with the spins in each pair constrained to point in opposite directions regardless of the applied magnetic field. As a result, the magnetic moments of such a pair of electrons exactly cancel one another. All that is left is the diamagnetism of the orbital motion which we have already explored. The vast majority of molecules are purely diamagnetic. A few molecules (really *very* few) contain an odd number of electrons. In such a molecule, total cancelation of spin moments in pairs is clearly impossible. Nitric oxide, NO, with 15 electrons in the molecule is an example; it is paramagnetic. The oxygen molecule O_2 contains 16 electrons, but its electronic structure happens to favor noncancelation of two of the electron spins. In single atoms the inner electrons are generally paired, and if there is an outer unpaired electron, its spin is often paired off with that of a neighbor when the atom is part of a compound or crystal. Certain atoms, however, do contain unpaired electron spins which remain relatively free to orient in a field even when the atom is packed in with others. Important examples are the elements ranging from chromium to copper in the periodic table, a sequence that includes iron, cobalt, and nickel. Another group of elements with this property is the rare earth sequence around gadolinium. Compounds or alloys of these elements are generally paramagnetic, and in some cases ferromagnetic. The number of free electron spins involved in paramagnetism is typically one or two per atom. We can think of each paramagnetic atom as equipped with one freely swiveling magnetic moment $\mathbf{m}$, which in a field $\mathbf{B}$ would be found pointing, like a tiny compass needle, in the direction of the field – if it were not for thermal disturbances.

Thermal agitation tends always to create a random distribution of spin axis directions. The degree of alignment that eventually prevails represents a compromise between the preference for the direction of lowest energy and the disorienting influence of thermal motion. We have met this problem before. In Section 10.12 we considered the alignment, by an electric field $\mathbf{E}$, of the electric dipole moments of polar molecules. It turned out to depend on the ratio of two energies: pE, the energetic advantage of orientation of a dipole moment $\mathbf{p}$ parallel to $\mathbf{E}$ as compared with an average over completely random orientations, and kT, the mean thermal energy associated with any form of molecular motion at absolute temperature T. Only if pE were much larger than kT would nearly complete alignment of the dipole moments be attained. If pE is much smaller than kT, the equilibrium polarization is equivalent to perfect alignment of a small fraction, approximately pE/kT, of the dipoles. We can take this result over directly for paramagnetism. We need only replace pE by mB, the energy involved in the orientation of a magnetic dipole moment $\mathbf{m}$ in a magnetic field $\mathbf{B}$. Providing mB/kT is small, it follows that the total magnetic moment, per unit volume, resulting from application of the field $\mathbf{B}$ to N dipoles per unit volume will be approximately

$$M \approx Nm\left(\frac{mB}{kT}\right) = \frac{Nm^2}{kT}B. \tag{11.50}$$

The induced moment is proportional to B and inversely proportional to the temperature.

For one electron spin moment ($m = 9.3 \cdot 10^{-24}$ joule/tesla) in our field of 1.8 tesla, mB is $1.7 \cdot 10^{-23}$ joule. For room temperature, kT is $4 \cdot 10^{-21}$ joule; in that case mB/kT is indeed small. But if we could lower the temperature to 1 K in the same field, mB/kT would be about unity. With further lowering of the temperature we could expect to approach complete alignment, with total moment approaching Nm. These conditions are quite frequently achieved in low-temperature experiments. Indeed, paramagnetism is both more impressive and more interesting at very low temperatures, in contrast to dielectric polarization. Molecular electric dipoles would be totally frozen in position, incapable of any reorientation. The electron spin moments are still remarkably free.

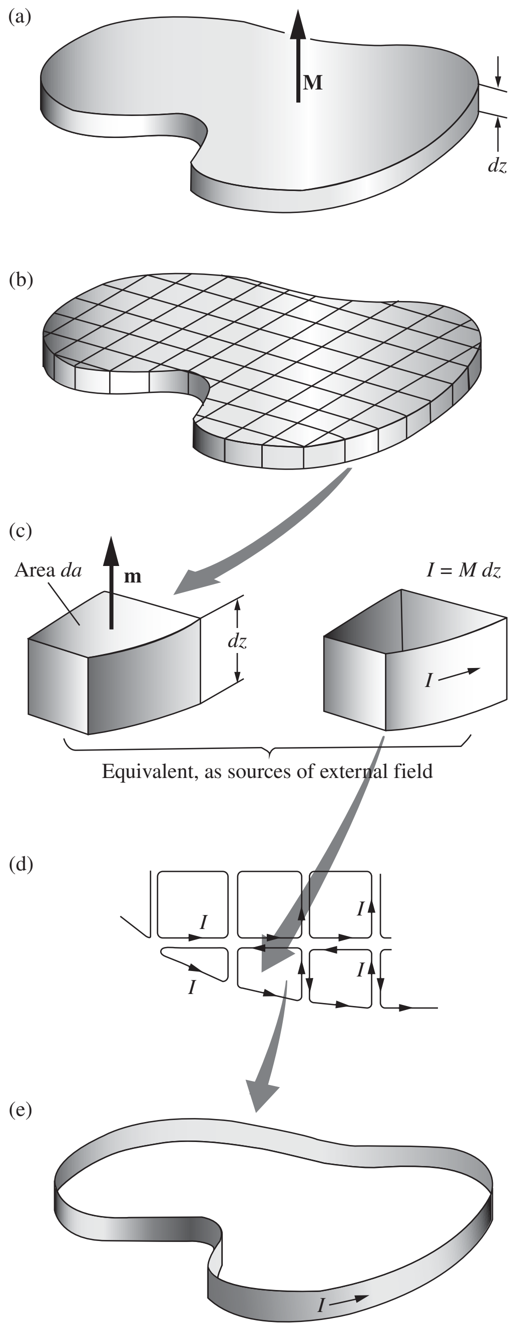


Figure 11.16.
The thin slab, magnetized perpendicular to its broad surface, is equivalent to a ribbon of current so far as its external field is concerned.

11.8 The magnetic field caused by magnetized matter

A block of material that contains, evenly distributed through its volume, a large number of atomic magnetic dipoles all pointing in the same direction, is said to be *uniformly magnetized.* The magnetization vector **M** is simply the product of the number of oriented dipoles per unit volume and the magnetic moment **m** of each dipole. We don't care how the alignment of these dipoles is maintained. There may be some field applied from another source, but we are not interested in that. We want to study only the field produced by the dipoles themselves.

Consider first a slab of material of thickness dz, sliced out perpendicular to the direction of magnetization, as shown in Fig. 11.16(a). The slab can be divided into little tiles, as indicated in Fig. 11.16(b). One such tile, which has a top surface of area da, contains a total dipole moment amounting to $M\,da\,dz$, since M is the dipole moment per unit volume. The magnetic field this tile produces at all *distant* points – distant compared with the size of the tile – is just that of any dipole with the same magnetic moment. We could construct a dipole of that strength by bending a conducting ribbon of width dz into the shape of the tile, and sending around this loop a current $I = M\,dz$; see Fig. 11.16(c). That will give the loop a dipole moment:

$$m = I \times \text{area} = (M\,dz)\,da, \tag{11.54}$$

which is the same as that of the tile.

Let us substitute such a current loop for every tile in the slab, as indicated in Fig. 11.16(d). The current is the same in all of these, and therefore, at every interior boundary we find equal and opposite currents, equivalent to zero current. Our "egg-crate" of loops is therefore equivalent to a single ribbon running around the outside, carrying the current $I = M\,dz$; see Fig. 11.16(e). Now, these tiles can be made quite small, so long as we don't subdivide all the way down to molecular size. They must be large enough so that their magnetization does not vary appreciably from one tile to the next. Within that limitation, we can state that the field at any *external* point, even close to the slab, is the same as that of the current ribbon.

It remains only to reconstruct a whole block from such laminations, or slabs, as in Fig. 11.17(a). The entire block is then equivalent to the

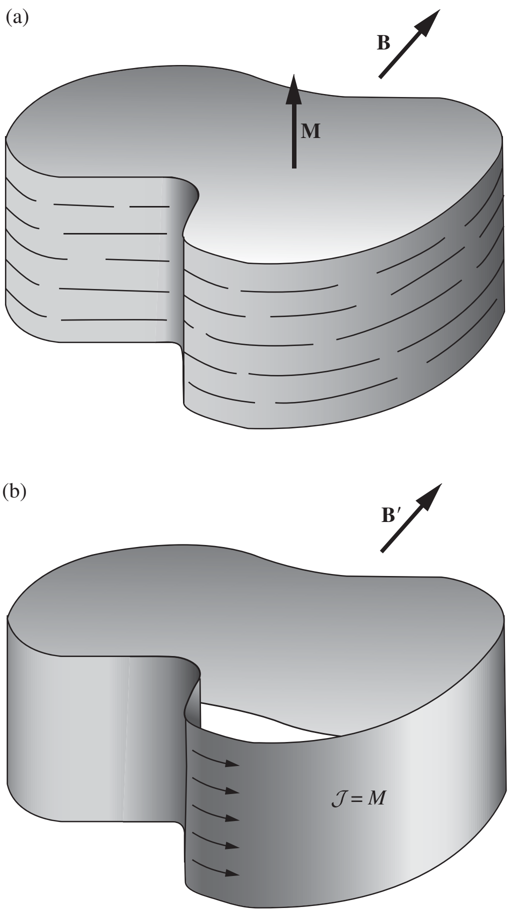


Figure 11.17.
A uniformly magnetized block is equivalent to a band of surface current.

wide ribbon in Fig. 11.17(b), around which flows a current $M\,dz$, in C/s, in every strip dz, or, stated more simply, a surface current of density $\mathcal{J}$, in C/(s-m), given by

$$\boxed{\mathcal{J} = M} \qquad (11.55)$$

The magnetic field **B** at any point outside the magnetized block in Fig. 11.17(a), and even close to the block provided we don't approach within molecular distances, is the same as the field **B**′ at the corresponding point in the neighborhood of the wide current ribbon in Fig. 11.17(b).

But what about the field inside the magnetized block? Here we face a question like the one we met in Chapter 10. Inside matter the magnetic field is not at all uniform if we observe it on the atomic scale, which we have been calling microscopic. It varies sharply in both magnitude and direction between points only a few angstroms apart. This *microscopic* field **B** is simply a magnetic field in vacuum, for from the microscopic viewpoint, as we emphasized in Chapter 10, matter is a collection of particles and electric charge in otherwise empty space. The only large-scale field that can be uniquely defined inside matter is the spatial average of the microscopic field.

Because of the absence of effects attributable to magnetic charge, we believe that the microscopic field itself satisfies $\text{div}\,\mathbf{B} = 0$. If that is true, it follows quite directly that the spatial average of the internal microscopic field in our block is equal to the field **B**′ inside the equivalent hollow cylinder of current.

To demonstrate this, consider the long rod uniformly magnetized parallel to its length, shown in Fig. 11.18(a). We have just shown that

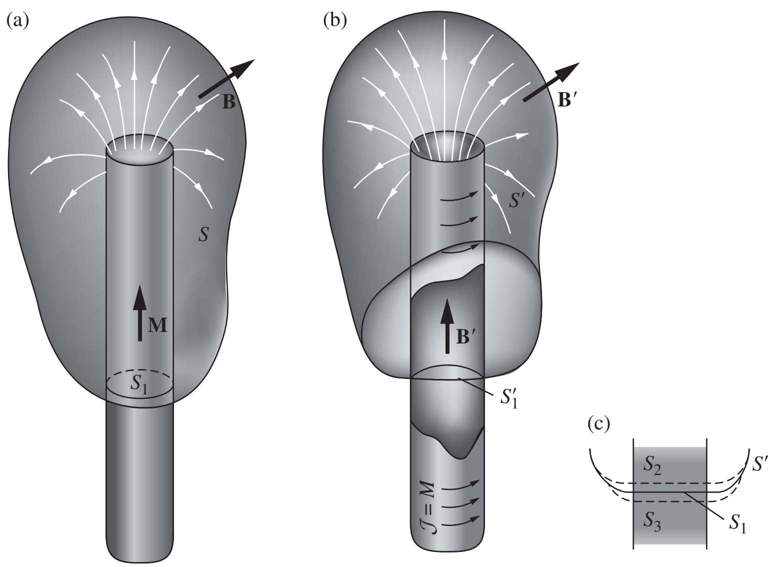


Figure 11.18.
(a) A uniformly magnetized cylindrical rod. (b) The equivalent hollow cylinder, or sheath, of current. Its field is **B**′. (c) We can sample the interior of the rod, and thus obtain a spatial average of the microscopic field, by closely spaced parallel surfaces, S_1, S_2,. . . .

the external field will be the same as that of the long cylinder of current (practically equivalent to a single-layer solenoid) shown in Fig. 11.18(b). The surface S in Fig. 11.18(a) indicates a closed surface that includes a portion S_1 passing through the interior of the rod. Because $\text{div}\,\mathbf{B} = 0$ for the internal microscopic field, as well as for the external field, $\text{div}\,\mathbf{B}$ is zero throughout the entire volume enclosed by S. It then follows from Gauss's theorem that the surface integral of $\mathbf{B}$ over S must be zero. The surface integral of $\mathbf{B}'$ over the closed surface S' in Fig. 11.18(b) is zero also. Over the portions of S and S' *external* to the cylinders, $\mathbf{B}$ and $\mathbf{B}'$ are identical. Therefore the surface integral of $\mathbf{B}$ over the internal disk S_1 must be equal to the surface integral of $\mathbf{B}'$ over the internal disk S_1'. This must hold also for any one of a closely spaced set of parallel disks, such as S_2, S_3, etc., indicated in Fig. 11.18(c), because the field outside the cylinder in this neighborhood is negligibly small, so that the outside parts don't change anything. Now, taking the surface integral over a series of equally spaced planes like this is a perfectly good way to compute the volume average of the field $\mathbf{B}$ in that neighborhood, for it samples all volume elements impartially. It follows that the spatial average of the microscopic field $\mathbf{B}$ inside the magnetized rod is equal to the field $\mathbf{B}'$ inside the current sheath of Fig. 11.18(b).

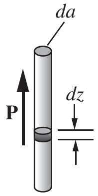
It is instructive to compare the arguments we have just developed with our analysis of the corresponding questions in Chapter 10. Figure 11.19 displays these developments side by side. You will see that they run logically parallel, but that at each stage there is a difference that reflects the essential asymmetry epitomized in the observation that *electric charges* are the source of *electric fields,* while *moving electric charges* are the source of *magnetic fields.* For example, in the arguments about the average of the microscopic field, the key to the problem in the electric case is the assumption that $\text{curl}\,\mathbf{E} = 0$ for the microscopic electric field. In the magnetic case, the key is the assumption that $\text{div}\,\mathbf{B} = 0$ for the microscopic magnetic field.

If the magnetization $\mathbf{M}$ within a volume of material is not uniform but instead varies with position as $\mathbf{M}(x, y, z)$, the equivalent current distribution is given simply by

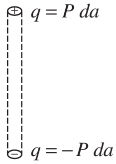
$$\boxed{\mathbf{J} = \text{curl}\,\mathbf{M}} \tag{11.56}$$

Let's see how this comes about in one situation. Suppose there is a magnetization in the z direction that gets stronger as we proceed in the y direction. This is represented in Fig. 11.20(a), which shows a small region in the material subdivided into little blocks. The blocks are supposed to be so small that we may consider the magnetization uniform within a single block. Then we can replace each block by a current ribbon, with surface current density $\mathcal{J} = M_z$. The current I carried by such a ribbon, if the block is Δz in height, is $\mathcal{J}\Delta z$ or $M_z \Delta z$. Now, each ribbon has a bit more

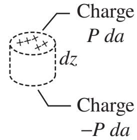
(a) As a source of *external* electric field **E**

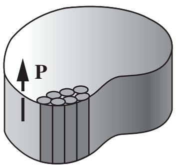


is equivalent to

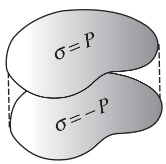


because a bit of polarized matter, volume $da \cdot dz$, has dipole moment equal to that of:



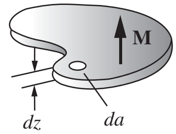


A uniformly polarized block can be subdivided into such rods. Hence the block has the same external field as two sheets of surface charge with $\sigma = P_n$

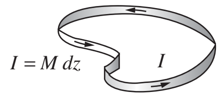


(More generally, for nonuniform polarization, polarized matter is equivalent to a charge distribution $\rho = -\text{div}\,\mathbf{P}$.)

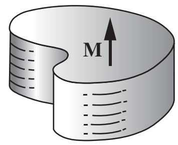
(b) As a source of *external* magnetic field **B**



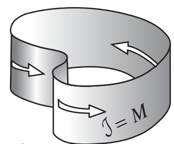
is equivalent to:



because a bit of magnetized matter, volume $da \cdot dz$, has dipole moment equal to that of:



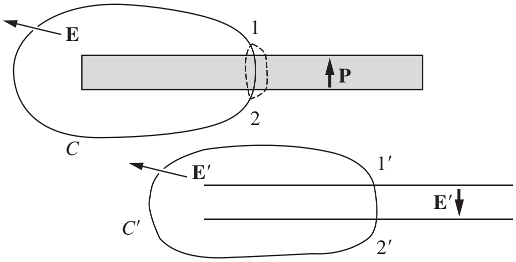
A uniformly magnetized block can be divided into such layers. Hence the block has the same external field as the wide ribbon of surface current with $\mathcal{J} = M$.



(More generally, for nonuniform magnetization, magnetized matter is equivalent to a current distribution $\mathbf{J} = \text{curl}\,\mathbf{M}$.)

PROOF THAT THE EQUIVALENCE EXTENDS TO
THE SPATIAL AVERAGE OF THE INTERNAL FIELDS

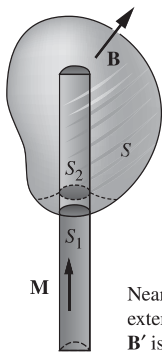
Consider a wide, thin, uniformly polarized slab and its equivalent sheets of surface charge.

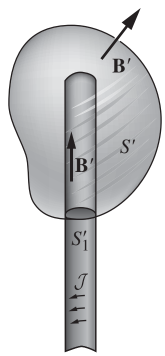


Near the middle the external field is slight and $\mathbf{E}'$ is uniform. If $\nabla \times \mathbf{E} = 0$ for the internal field, then $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$. But $\mathbf{E} = \mathbf{E}'$ on the external path. Hence $\int_1^2 \mathbf{E} \cdot d\mathbf{l} = \int_{1'}^{2'} \mathbf{E}' \cdot d\mathbf{l}'$ for all internal paths.

Conclusion: $\langle \mathbf{E} \rangle = \mathbf{E}'$; the spatial average of the internal electric field is equal to the field $\mathbf{E}'$ that would be produced at that point in empty space by the equivalent charge distribution described above (together with any external sources).

Consider a long uniformly magnetized column and its equivalent cylinder of surface current.





Near the middle the external field is slight and $\mathbf{B}'$ is uniform. If $\nabla \cdot \mathbf{B} = 0$ for the internal field, then $\int_S \mathbf{B} \cdot d\mathbf{a} = 0$. But $\mathbf{B} = \mathbf{B}'$ on the surface external to the column. Hence $\int_{S_1} \mathbf{B} \cdot d\mathbf{a} = \int_{S_1'} \mathbf{B}' \cdot d\mathbf{a}'$ over any interior portion of surface, like S_1, S_2, etc.

Conclusion: $\langle \mathbf{B} \rangle = \mathbf{B}'$; the spatial average of the internal magnetic field is equal to the field $\mathbf{B}'$ that would be produced at that point in empty space by the equivalent charge distribution described above (together with any external sources).

Figure 11.19.
The electric (a) and magnetic (b) cases compared.

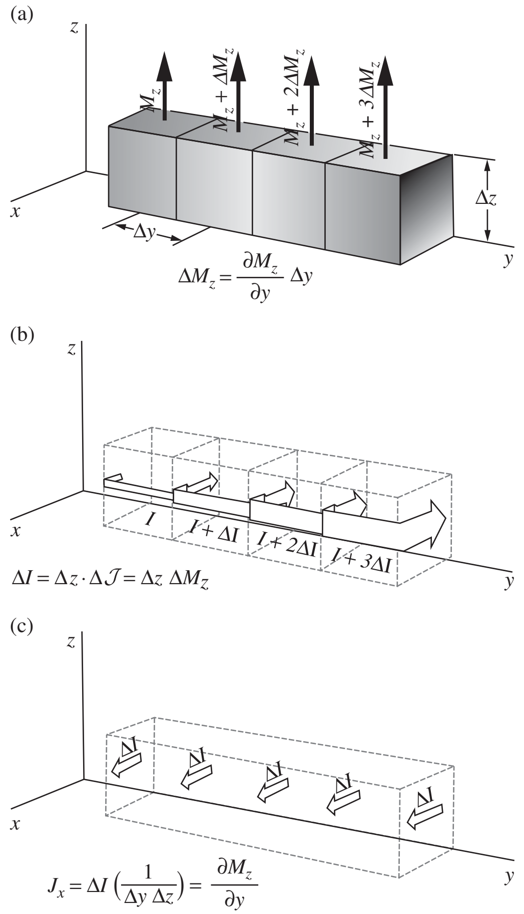


Figure 11.20.
Nonuniform magnetization is equivalent to a volume current density.

current density than the one to the left of it. The current in each loop is greater than the current in the loop to the left by

$$\Delta I = \Delta z\, \Delta M_z = \Delta z \frac{\partial M_z}{\partial y}\, \Delta y. \tag{11.57}$$

At every interface in this row of blocks there is a net current in the x direction of magnitude ΔI; see Fig. 11.20(c). To get the current per unit area flowing in the x direction we have to multiply by the number of blocks per unit area, which is $1/(\Delta y\, \Delta z)$. Thus

$$J_x = \Delta I \left(\frac{1}{\Delta y\, \Delta z}\right) = \frac{\partial M_z}{\partial y}. \tag{11.58}$$

Another way of getting an x-directed current is to have a y component of magnetization that varies in the z direction. If you trace through that case, using a vertical column of blocks, you will find that the net x-directed current density is given by

$$J_x = -\frac{\partial M_y}{\partial z}. \tag{11.59}$$

In general then, by superposition of these two situations,

$$J_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z} = (\text{curl}\,\mathbf{M})_x, \tag{11.60}$$

which is enough to establish Eq. (11.56). In Section 11.10 we will relabel the $\mathbf{J}$ in Eq. (11.56) as $\mathbf{J}_{\text{bound}}$, because it arises from the orbital and spin angular momentum of electrons within atoms. The present $\mathbf{J}_{\text{bound}} = \text{curl}\,\mathbf{M}$ result for a magnetized material then clearly parallels the $-\rho_{\text{bound}} = \text{div}\,\mathbf{P}$ result for a polarized material that we derived in Eq. (10.61).

Example Show that the $\mathcal{J} = M$ result for the magnetized slab in Fig. 11.17 follows from $\mathbf{J} = \text{curl}\,\mathbf{M}$, by integrating $\mathbf{J} = \text{curl}\,\mathbf{M}$ over an appropriate area. What about the more general case where $\mathbf{M}$ isn't parallel to the boundary of the slab?

Solution Consider a thin rectangle, with one of its long sides inside the material and the other outside, as in Fig. 11.21(a). If we integrate $\mathbf{J} = \text{curl}\,\mathbf{M}$ over the surface S of this rectangle, we can use Stokes' theorem to write

$$\int_S \mathbf{J}\cdot d\mathbf{a} = \int_S \text{curl}\,\mathbf{M}\cdot d\mathbf{a} \implies I_S = \int_C \mathbf{M}\cdot d\mathbf{s}, \tag{11.61}$$

where I_S is the current passing through S. This current can be written as $\mathcal{J}\ell$, where ℓ is the height of the rectangle. This is true because we can make the rectangle arbitrarily thin, so any current passing through it must arise from a surface current density $\mathcal{J}$. The integral $\int_C \mathbf{M}\cdot d\mathbf{s}$ simply equals $M\ell$ (if the integral runs around the loop in the clockwise direction), because $\mathbf{M}$ is nonzero only along the left side of the rectangle. Equation (11.61) therefore gives $\mathcal{J}\ell = M\ell \implies \mathcal{J} = M$, as desired. If $\mathbf{M}$ points upward, the surface current density flows into the page.

If we have the more general case shown in Fig. 11.21(b), where the surface is tilted with respect to the direction of $\mathbf{M}$, then integrating over the area of the thin rectangle still gives $I_S = \int_C \mathbf{M}\cdot d\mathbf{s}$. But now only the component of $\mathbf{M}$ parallel to the long side of the rectangle survives in the dot product. Call this component $M_\parallel$. The above reasoning then quickly yields $\mathcal{J} = M_\parallel$. (Compare this with the surface charge density $\sigma = P_\perp$ for an electrically polarized material.) You can also arrive at this result by taking into account all the tiny current loops, as we did in Fig. 11.16. In short, the same number of current loops fit into a given height, but the relevant surface area is larger if the surface is tilted. So the surface current density is smaller.

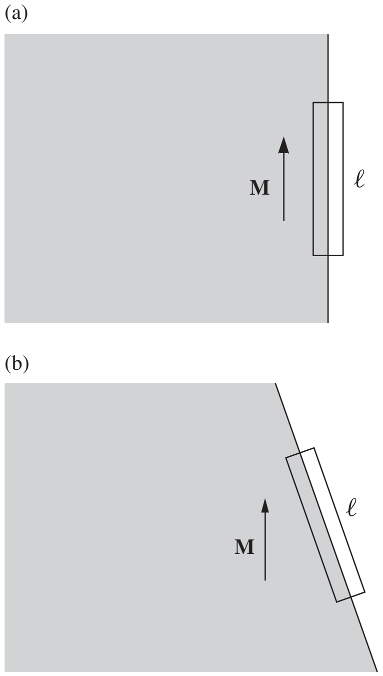


Figure 11.21.
The surface integral over a thin rectangle at the boundary, combined with Stokes' theorem, shows that $\mathcal{J} = M$ follows from $\mathbf{J} = \text{curl}\,\mathbf{M}$.

CHAPTER 6

Magnetic Fields in Matter

6.1 ■ MAGNETIZATION

6.1.1 ■ Diamagnets, Paramagnets, Ferromagnets

If you ask the average person what “magnetism” is, you will probably be told about refrigerator decorations, compass needles, and the North Pole—none of which has any obvious connection with moving charges or current-carrying wires. Yet all magnetic phenomena are due to electric charges in motion, and in fact, if you could examine a piece of magnetic material on an atomic scale you *would* find tiny currents: electrons orbiting around nuclei and spinning about their axes. For macroscopic purposes, these current loops are so small that we may treat them as magnetic dipoles. Ordinarily, they cancel each other out because of the random orientation of the atoms. But when a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or **magnetized**.

Unlike electric polarization, which is almost always in the same direction as **E**, some materials acquire a magnetization *parallel* to **B** (**paramagnets**) and some *opposite* to **B** (**diamagnets**). A few substances (called **ferromagnets**, in deference to the most common example, iron) retain their magnetization even after the external field has been removed—for these, the magnetization is not determined by the *present* field but by the whole magnetic “history” of the object. Permanent magnets made of iron are the most familiar examples of magnetism, but from a theoretical point of view they are the most complicated; I’ll save ferromagnetism for the end of the chapter, and begin with qualitative models of paramagnetism and diamagnetism.

6.1.2 ■ Torques and Forces on Magnetic Dipoles

A magnetic dipole experiences a torque in a magnetic field, just as an electric dipole does in an electric field. Let’s calculate the torque on a rectangular current loop in a uniform field **B**. (Since any current loop could be built up from infinitesimal rectangles, with all the “internal” sides canceling, as indicated in Fig. 6.1, there is no real loss of generality here; but if you prefer to start from scratch with an arbitrary shape, see Prob. 6.2.) Center the loop at the origin, and tilt it an angle θ from the z axis towards the y axis (Fig. 6.2). Let **B** point in the z direction. The forces on the two sloping sides cancel (they tend to *stretch* the loop, but they don’t

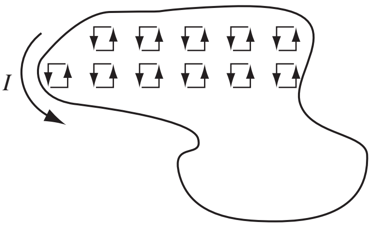


FIGURE 6.1

rotate it). The forces on the “horizontal” sides are likewise equal and opposite (so the net *force* on the loop is zero), but they do generate a torque:

$$\mathbf{N} = aF \sin\theta\,\hat{\mathbf{x}}.$$

The magnitude of the force on each of these segments is

$$F = IbB,$$

and therefore

$$\mathbf{N} = IabB \sin\theta\,\hat{\mathbf{x}} = mB \sin\theta\,\hat{\mathbf{x}},$$

or

$$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B},} \tag{6.1}$$

where $m = Iab$ is the magnetic dipole moment of the loop. Equation 6.1 gives the torque on any localized current distribution, in the presence of a *uniform* field; in a *nonuniform* field it is the exact torque (about the center) for a *perfect* dipole of infinitesimal size.

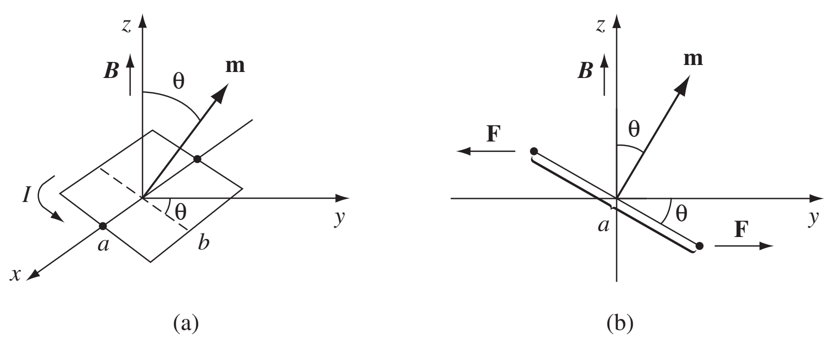


FIGURE 6.2

Notice that Eq. 6.1 is identical in form to the electrical analog, Eq. 4.4: $\mathbf{N} = \mathbf{p} \times \mathbf{E}$. In particular, the torque is again in such a direction as to line the dipole up *parallel* to the field. It is this torque that accounts for **paramagnetism**. Since every electron constitutes a magnetic dipole (picture it, if you wish, as a tiny spinning sphere of charge), you might expect paramagnetism to be a universal phenomenon. Actually, quantum mechanics (specifically, the Pauli exclusion principle) tends to lock the electrons within a given atom together in pairs with opposing spins,[1] and this effectively neutralizes the torque on the combination. As a result, paramagnetism most often occurs in atoms or molecules with an odd number of electrons, where the "extra" unpaired member is subject to the magnetic torque. Even here, the alignment is far from complete, since random thermal collisions tend to destroy the order.

In a uniform field, the net *force* on a current loop is zero:

$$\mathbf{F} = I \oint (d\mathbf{l} \times \mathbf{B}) = I \left(\oint d\mathbf{l} \right) \times \mathbf{B} = \mathbf{0};$$

the constant $\mathbf{B}$ comes outside the integral, and the net displacement $\oint d\mathbf{l}$ around a closed loop vanishes. In a *nonuniform* field this is no longer the case. For example, suppose a circular wire ring of radius R, carrying a current I, is suspended above a short solenoid in the "fringing" region (Fig. 6.3). Here $\mathbf{B}$ has a radial component, and there is a net downward force on the loop (Fig. 6.4):

$$F = 2\pi I R B \cos\theta. \tag{6.2}$$

For an *infinitesimal* loop, with dipole moment $\mathbf{m}$, in a field $\mathbf{B}$, the force is

$$\boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})} \tag{6.3}$$

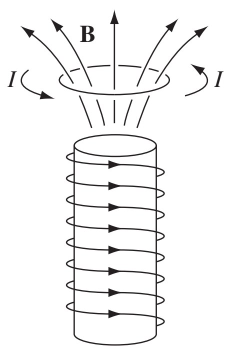


FIGURE 6.3

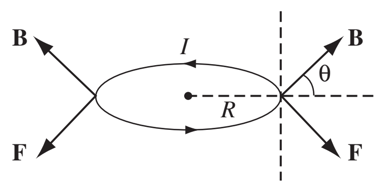
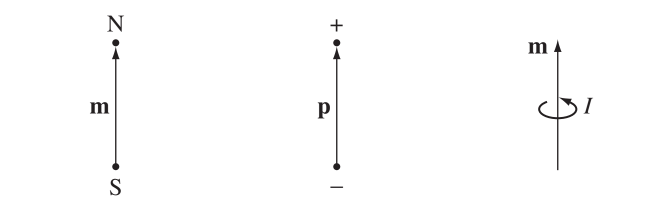


FIGURE 6.4

[1]This is not always true for the outermost electrons in unfilled shells.

(see Prob. 6.4). Once again the magnetic formula is identical to its electrical "twin," if we write the latter in the form $\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E})$. (See footnote to Eq. 4.5.)

If you're starting to get a sense of *déjà vu*, perhaps you will have more respect for those early physicists who thought magnetic dipoles consisted of positive and negative magnetic "charges" (north and south "poles," they called them), separated by a small distance, just like electric dipoles (Fig. 6.5(a)). They wrote down a "Coulomb's law" for the attraction and repulsion of these poles, and developed the whole of magnetostatics in exact analogy to electrostatics. It's not a bad model, for many purposes—it gives the correct field of a dipole (at least, away from the origin), the right torque on a dipole (at least, on a *stationary* dipole), and the proper force on a dipole (at least, in the absence of external currents). But it's bad physics, because *there's no such thing* as a single magnetic north pole or south pole. If you break a bar magnet in half, you don't get a north pole in one hand and a south pole in the other; you get two complete magnets. Magnetism is *not* due to magnetic monopoles, but rather to *moving electric charges*; magnetic dipoles are tiny current loops (Fig. 6.5(c)), and it's an extraordinary thing, really, that the formulas involving **m** bear any resemblance to the corresponding formulas for **p**. Sometimes it is easier to think in terms of the "Gilbert" model of a magnetic dipole (separated monopoles), instead of the physically correct "Ampère" model (current loop). Indeed, this picture occasionally offers a quick and clever solution to an otherwise cumbersome problem (you just copy the corresponding result from electrostatics, changing **p** to **m**, $1/\epsilon_0$ to μ_0, and **E** to **B**). But whenever the *close-up* features of the dipole come into play, the two models can yield strikingly different answers. My advice is to use the Gilbert model, if you like, to get an intuitive "feel" for a problem, but never rely on it for quantitative results.



(a) Magnetic dipole (Gilbert model) (b) Electric dipole (c) Magnetic dipole (Ampère model)

FIGURE 6.5

Problem 6.1 Calculate the torque exerted on the square loop shown in Fig. 6.6, due to the circular loop (assume r is much larger than a or b). If the square loop is free to rotate, what will its equilibrium orientation be?

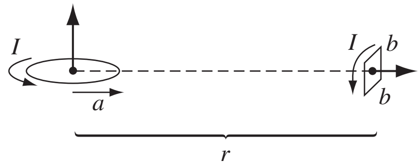


FIGURE 6.6

Problem 6.2 Starting from the Lorentz force law, in the form of Eq. 5.16, show that the torque on *any* steady current distribution (not just a square loop) in a uniform field $\mathbf{B}$ is $\mathbf{m} \times \mathbf{B}$.

Problem 6.3 Find the force of attraction between two magnetic dipoles, $\mathbf{m}_1$ and $\mathbf{m}_2$, oriented as shown in Fig. 6.7, a distance r apart, (a) using Eq. 6.2, and (b) using Eq. 6.3.

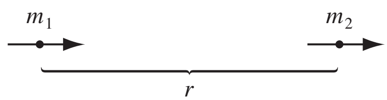


FIGURE 6.7

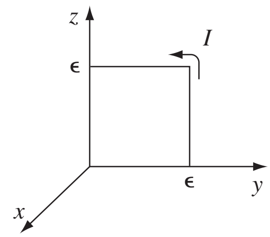


FIGURE 6.8

Problem 6.4 Derive Eq. 6.3. [Here's one way to do it: Assume the dipole is an infinitesimal square, of side ϵ (if it's not, chop it up into squares, and apply the argument to each one). Choose axes as shown in Fig. 6.8, and calculate $\mathbf{F} = I \int (d\mathbf{l} \times \mathbf{B})$ along each of the four sides. Expand $\mathbf{B}$ in a Taylor series—on the right side, for instance,

$$\mathbf{B} = \mathbf{B}(0, \epsilon, z) \cong \mathbf{B}(0, 0, z) + \epsilon \left.\frac{\partial \mathbf{B}}{\partial y}\right|_{(0,0,z)}.$$

For a more sophisticated method, see Prob. 6.22.]

Problem 6.5 A uniform current density $\mathbf{J} = J_0\,\hat{\mathbf{z}}$ fills a slab straddling the yz plane, from $x = -a$ to $x = +a$. A magnetic dipole $\mathbf{m} = m_0\,\hat{\mathbf{x}}$ is situated at the origin.

(a) Find the force on the dipole, using Eq. 6.3.

(b) Do the same for a dipole pointing in the y direction: $\mathbf{m} = m_0\hat{\mathbf{y}}$.

(c) In the *electrostatic* case, the expressions $\mathbf{F} = \nabla(\mathbf{p} \cdot \mathbf{E})$ and $\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}$ are equivalent (prove it), but this is *not* the case for the magnetic analogs (explain why). As an example, calculate $(\mathbf{m} \cdot \nabla)\mathbf{B}$ for the configurations in (a) and (b).

6.1.3 ■ Effect of a Magnetic Field on Atomic Orbits

Electrons not only *spin*; they also *revolve* around the nucleus—for simplicity, let's assume the orbit is a circle of radius R (Fig. 6.9). Although technically this orbital motion does not constitute a steady current, in practice the period $T = 2\pi R/v$ is so short that unless you blink awfully fast, it's going to *look* like a steady current:

$$I = \frac{-e}{T} = -\frac{ev}{2\pi R}.$$

(The minus sign accounts for the negative charge of the electron.) Accordingly, the orbital dipole moment ($I\pi R^2$) is

$$\mathbf{m} = -\frac{1}{2}evR\,\hat{\mathbf{z}}. \tag{6.4}$$

Like any other magnetic dipole, this one is subject to a torque ($\mathbf{m} \times \mathbf{B}$) when you turn on a magnetic field. But it's a lot harder to tilt the entire orbit than it is the spin, so the orbital contribution to paramagnetism is small. There is, however, a more significant effect on the orbital motion: The electron *speeds up* or *slows down*, depending on the orientation of $\mathbf{B}$. For whereas the centripetal acceleration v^2/R is ordinarily sustained by electrical forces alone,[2]

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} = m_e\frac{v^2}{R}, \tag{6.5}$$

in the presence of a magnetic field there is an additional force, $-e(\mathbf{v} \times \mathbf{B})$. For the sake of argument, let's say that $\mathbf{B}$ is perpendicular to the plane of the orbit, as shown in Fig. 6.10; then

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{R^2} + e\bar{v}B = m_e\frac{\bar{v}^2}{R}. \tag{6.6}$$

Under these conditions, the new speed $\bar{v}$ is *greater* than v:

$$e\bar{v}B = \frac{m_e}{R}(\bar{v}^2 - v^2) = \frac{m_e}{R}(\bar{v} + v)(\bar{v} - v),$$

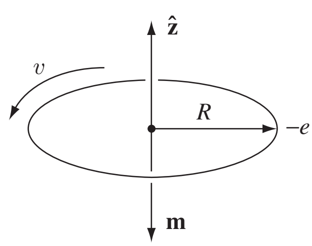


FIGURE 6.9

[2]To avoid confusion with the magnetic dipole moment m, I'll write the electron mass with subscript: m_e.

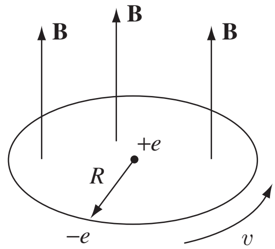


FIGURE 6.10

or, assuming the change $\Delta v = \bar{v} - v$ is small,

$$\Delta v = \frac{eRB}{2m_e}. \tag{6.7}$$

When **B** is turned on, then, the electron speeds up.[3]

A change in orbital speed means a change in the dipole moment (Eq. 6.4):

$$\Delta\mathbf{m} = -\frac{1}{2}e(\Delta v)R\,\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}. \tag{6.8}$$

Notice that *the change in* **m** *is opposite to the direction of* **B**. (An electron circling the other way would have a dipole moment pointing upward, but such an orbit would be slowed down by the field, so the *change* is still opposite to **B**.) Ordinarily, the electron orbits are randomly oriented, and the orbital dipole moments cancel out. But in the presence of a magnetic field, each atom picks up a little "extra" dipole moment, and these increments are all *antiparallel* to the field. This is the mechanism responsible for **diamagnetism**. It is a universal phenomenon, affecting all atoms. However, it is typically much weaker than paramagnetism, and is therefore observed mainly in atoms with *even* numbers of electrons, where paramagnetism is usually absent.

In deriving Eq. 6.8, I assumed that the orbit remains circular, with its original radius R. I cannot offer a justification for this at the present stage. If the atom is stationary while the field is turned on, then my assumption can be proved—this is not magneto*statics*, however, and the details will have to await Chapter 7 (see Prob. 7.52). If the atom is moved into the field, the situation is enormously more complicated. But never mind—I'm only trying to give you a qualitative account of diamagnetism. Assume, if you prefer, that the velocity remains the same while the *radius* changes—the formula (Eq. 6.8) is altered (by a factor of 2), but the qualitative conclusion is unaffected. The truth is that this classical model is fundamentally flawed (diamagnetism is really a *quantum* phenomenon), so there's

[3]I said (Eq. 5.11) that magnetic fields do no work, and are incapable of speeding a particle up. I stand by that. However, as we shall see in Chapter 7, a *changing* magnetic field induces an *electric* field, and it is the latter that accelerates the electrons in this instance.

not much point in refining the details.[4] What *is* important is the empirical fact that in diamagnetic materials the induced dipole moments point opposite to the magnetic field.

6.1.4 ■ Magnetization

In the presence of a magnetic field, matter becomes *magnetized*; that is, upon microscopic examination, it will be found to contain many tiny dipoles, with a net alignment along some direction. We have discussed two mechanisms that account for this magnetic polarization: (1) paramagnetism (the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field) and (2) diamagnetism (the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field). Whatever the cause, we describe the state of magnetic polarization by the vector quantity

$$\mathbf{M} \equiv \textit{magnetic dipole moment per unit volume.} \tag{6.9}$$

M is called the **magnetization**; it plays a role analogous to the polarization **P** in electrostatics. In the following section, we will not worry about how the magnetization *got* there—it could be paramagnetism, diamagnetism, or even ferromagnetism—we shall take **M** as *given*, and calculate the field this magnetization itself produces.

Incidentally, it may have surprised you to learn that materials other than the famous ferromagnetic trio (iron, nickel, and cobalt) are affected by a magnetic field *at all*. You cannot, of course, pick up a piece of wood or aluminum with a magnet. The reason is that diamagnetism and paramagnetism are extremely weak: It takes a delicate experiment and a powerful magnet to detect them at all. If you were to suspend a piece of paramagnetic material above a solenoid, as in Fig. 6.3, the induced magnetization would be upward, and hence the force downward. By contrast, the magnetization of a diamagnetic object would be downward and the force upward. In general, when a sample is placed in a region of nonuniform field, the *paramagnet is attracted into the field*, whereas the *diamagnet is repelled away*. But the actual forces are pitifully weak—in a typical experimental arrangement the force on a comparable sample of iron would be 10^4 or 10^5 times as great. That's why it was reasonable for us to calculate the field inside a piece of copper wire, say, in Chapter 5, without worrying about the effects of magnetization.[5]

[4] S. L. O'Dell and R. K. P. Zia, *Am. J. Phys.* **54**, 32, (1986); R. Peierls, *Surprises in Theoretical Physics*, Section 4.3 (Princeton, N.J.: Princeton University Press, 1979); R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Vol. 2, Sec. 34–36 (New York: Addison-Wesley, 1966).

[5] In 1997 Andre Geim managed to levitate a live frog (diamagnetic) for 30 minutes; he was awarded the 2000 Ig Nobel prize for this achievement, and later (2010) the Nobel prize for research on graphene. See M. V. Berry and A. K. Geim, *Eur. J. Phys.* **18**, 307 (1997) and Geim, *Physics Today*, September 1998, p. 36.

Problem 6.6 Of the following materials, which would you expect to be paramagnetic and which diamagnetic: aluminum, copper, copper chloride ($CuCl_2$), carbon, lead, nitrogen (N_2), salt (NaCl), sodium, sulfur, water? (Actually, copper is slightly *dia*magnetic; otherwise they're all what you'd expect.)

6.2 ■ THE FIELD OF A MAGNETIZED OBJECT

6.2.1 ■ Bound Currents

Suppose we have a piece of magnetized material; the magnetic dipole moment per unit volume, **M**, is given. What field does this object produce? Well, the vector potential of a single dipole **m** is given by Eq. 5.85:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{6.10}$$

In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\mathbf{M}\,d\tau'$, so the total vector potential is (Fig. 6.11)

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{r}')\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'. \tag{6.11}$$

That *does* it, in principle. But, as in the electrical case (Sect. 4.2.1), the integral can be cast in a more illuminating form by exploiting the identity

$$\nabla'\frac{1}{\mathscr{r}} = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.$$

With this,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\left[\mathbf{M}(\mathbf{r}')\times\left(\nabla'\frac{1}{\mathscr{r}}\right)\right]d\tau'.$$

Integrating by parts, using product rule 7, gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left\{\int\frac{1}{\mathscr{r}}[\nabla'\times\mathbf{M}(\mathbf{r}')]\,d\tau' - \int\nabla'\times\left[\frac{\mathbf{M}(\mathbf{r}')}{\mathscr{r}}\right]d\tau'\right\}.$$

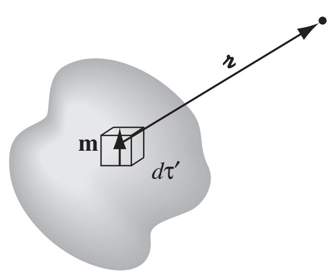


FIGURE 6.11

Problem 1.61(b) invites us to express the latter as a surface integral,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{1}{\mathscr{r}}[\nabla' \times \mathbf{M}(\mathbf{r}')]\,d\tau' + \frac{\mu_0}{4\pi}\oint \frac{1}{\mathscr{r}}[\mathbf{M}(\mathbf{r}') \times d\mathbf{a}']. \tag{6.12}$$

The first term looks just like the potential of a volume current,

$$\boxed{\mathbf{J}_b = \nabla \times \mathbf{M},} \tag{6.13}$$

while the second looks like the potential of a surface current,

$$\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}},} \tag{6.14}$$

where $\hat{\mathbf{n}}$ is the normal unit vector. With these definitions,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da'. \tag{6.15}$$

What this means is that the potential (and hence also the field) of a magnetized object is the same as would be produced by a volume current $\mathbf{J}_b = \nabla \times \mathbf{M}$ throughout the material, plus a surface current $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$, on the boundary. Instead of integrating the contributions of all the infinitesimal dipoles, using Eq. 6.11, we first determine the **bound currents**, and then find the field *they* produce, in the same way we would calculate the field of any other volume and surface currents. Notice the striking parallel with the electrical case: there the field of a polarized object was the same as that of a bound volume charge $\rho_b = -\nabla \cdot \mathbf{P}$ plus a bound surface charge $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$.

Example 6.1. Find the magnetic field of a uniformly magnetized sphere.

Solution

Choosing the z axis along the direction of $\mathbf{M}$ (Fig. 6.12), we have

$$\mathbf{J}_b = \nabla \times \mathbf{M} = \mathbf{0}, \quad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta\,\hat{\boldsymbol{\phi}}.$$

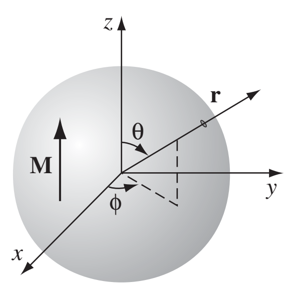


FIGURE 6.12

Now, a rotating spherical shell, of uniform surface charge σ, corresponds to a surface current density

$$\mathbf{K} = \sigma \mathbf{v} = \sigma \omega R \sin\theta \,\hat{\boldsymbol{\phi}}.$$

It follows, therefore, that the field of a uniformly magnetized sphere is identical to the field of a spinning spherical shell, with the identification $\sigma R\boldsymbol{\omega} \to \mathbf{M}$. Referring back to Ex. 5.11, I conclude that

$$\mathbf{B} = \frac{2}{3}\mu_0 \mathbf{M}, \tag{6.16}$$

inside the sphere, while the field outside is the same as that of a perfect dipole,

$$\mathbf{m} = \frac{4}{3}\pi R^3 \mathbf{M}.$$

Notice that the internal field is *uniform*, like the electric field inside a uniformly polarized sphere (Eq. 4.14), although the actual *formulas* for the two cases are curiously different $\left(\frac{2}{3}\text{ in place of } -\frac{1}{3}\right)$.[6] The external fields are also analogous: pure dipole in both instances.

Problem 6.7 An infinitely long circular cylinder carries a uniform magnetization $\mathbf{M}$ parallel to its axis. Find the magnetic field (due to $\mathbf{M}$) inside and outside the cylinder.

Problem 6.8 A long circular cylinder of radius R carries a magnetization $\mathbf{M} = ks^2\,\hat{\boldsymbol{\phi}}$, where k is a constant, s is the distance from the axis, and $\hat{\boldsymbol{\phi}}$ is the usual azimuthal unit vector (Fig. 6.13). Find the magnetic field due to $\mathbf{M}$, for points inside and outside the cylinder.

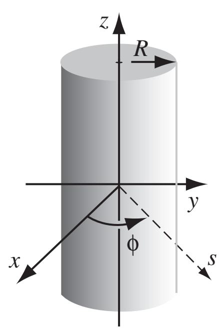


FIGURE 6.13

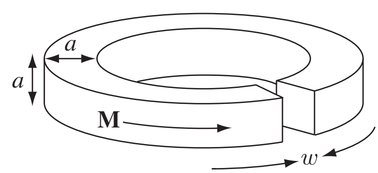


FIGURE 6.14

[6]It is no accident that the same factors appear in the "contact" term for the fields of electric and magnetic dipoles (Eqs. 3.106 and 5.94). In fact, one good way to model a perfect dipole is to take the limit ($R \to 0$) of a polarized/magnetized sphere.

Problem 6.9 A short circular cylinder of radius a and length L carries a "frozen-in" uniform magnetization $\mathbf{M}$ parallel to its axis. Find the bound current, and sketch the magnetic field of the cylinder. (Make three sketches: one for $L \gg a$, one for $L \ll a$, and one for $L \approx a$.) Compare this **bar magnet** with the bar electret of Prob. 4.11.

Problem 6.10 An iron rod of length L and square cross section (side a) is given a uniform longitudinal magnetization $\mathbf{M}$, and then bent around into a circle with a narrow gap (width w), as shown in Fig. 6.14. Find the magnetic field at the center of the gap, assuming $w \ll a \ll L$. [*Hint:* treat it as the superposition of a *complete* torus plus a square loop with reversed current.]

6.2.2 ■ Physical Interpretation of Bound Currents

In the last section, we found that the field of a magnetized object is identical to the field that would be produced by a certain distribution of "bound" currents, $\mathbf{J}_b$ and $\mathbf{K}_b$. I want to show you how these bound currents arise physically. This will be a heuristic argument—the rigorous derivation has already been given. Figure 6.15 depicts a thin slab of uniformly magnetized material, with the dipoles represented by tiny current loops. Notice that all the "internal" currents cancel: every time there is one going to the right, a contiguous one is going to the left. However, at the edge there is *no adjacent loop to do the canceling*. The whole thing, then, is equivalent to a single ribbon of current I flowing around the boundary (Fig. 6.16).

What *is* this current, in terms of $\mathbf{M}$? Say that each of the tiny loops has area a and thickness t (Fig. 6.17). In terms of the magnetization M, its dipole moment

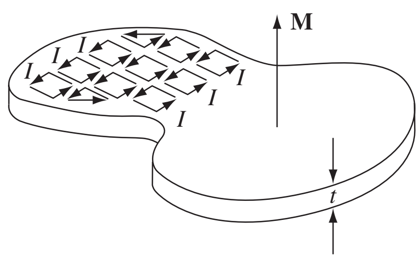


FIGURE 6.15

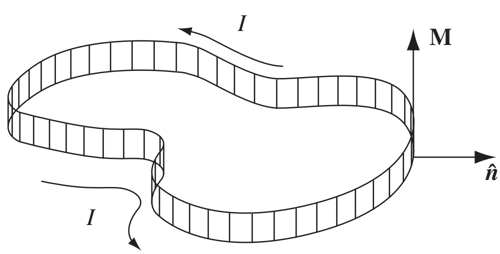


FIGURE 6.16

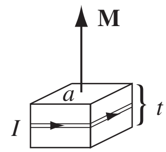


FIGURE 6.17

is $m = Mat$. In terms of the circulating current I, however, $m = Ia$. Therefore $I = Mt$, so the surface current is $K_b = I/t = M$. Using the outward-drawn unit vector $\hat{\mathbf{n}}$ (Fig. 6.16), the direction of $\mathbf{K}_b$ is conveniently indicated by the cross product:

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}.$$

(This expression also records the fact that there is *no* current on the top or bottom surface of the slab; here $\mathbf{M}$ is parallel to $\hat{\mathbf{n}}$, so the cross product vanishes.)

This bound surface current is exactly what we obtained in Sect. 6.2.1. It is a peculiar *kind* of current, in the sense that no single charge makes the whole trip—on the contrary, each charge moves only in a tiny little loop within a single atom. Nevertheless, the net effect is a macroscopic current flowing over the surface of the magnetized object. We call it a "bound" current to remind ourselves that every charge is attached to a particular atom, but it's a perfectly genuine current, and it produces a magnetic field in the same way any other current does.

When the magnetization is *non*uniform, the internal currents no longer cancel. Figure 6.18(a) shows two adjacent chunks of magnetized material, with a larger arrow on the one to the right suggesting greater magnetization at that point. On the surface where they join, there is a net current in the x direction, given by

$$I_x = [M_z(y + dy) - M_z(y)]\,dz = \frac{\partial M_z}{\partial y}\,dy\,dz.$$

The corresponding volume current density is therefore

$$(J_b)_x = \frac{\partial M_z}{\partial y}.$$

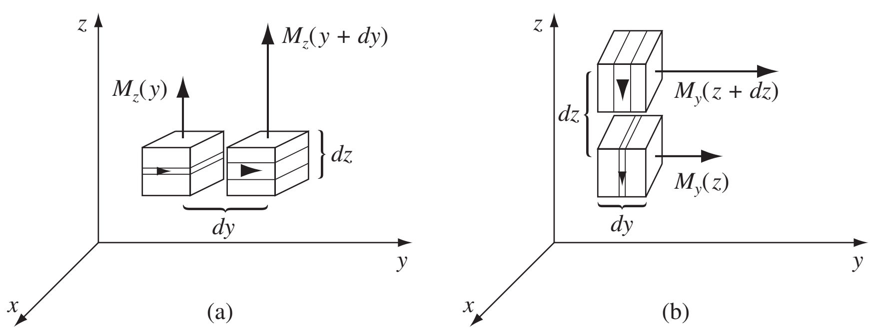


FIGURE 6.18

By the same token, a nonuniform magnetization in the y direction would contribute an amount $-\partial M_y/\partial z$ (Fig. 6.18(b)), so

$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}.$$

In general, then,

$$\mathbf{J}_b = \nabla \times \mathbf{M},$$

consistent, again, with the result of Sect. 6.2.1.

Incidentally, like any other steady current, $\mathbf{J}_b$ should obey the conservation law 5.33:

$$\nabla \cdot \mathbf{J}_b = 0.$$

Does it? *Yes*, for the divergence of a curl is *always* zero.

6.2.3 ■ The Magnetic Field Inside Matter

Like the electric field, the actual *microscopic* magnetic field inside matter fluctuates wildly from point to point and instant to instant. When we speak of "the" magnetic field in matter, we mean the *macroscopic* field: the average over regions large enough to contain many atoms. (The magnetization **M** is "smoothed out" in the same sense.) It is this macroscopic field that one obtains when the methods of Sect. 6.2.1 are applied to points inside magnetized material, as you can prove for yourself in the following problem.

Problem 6.11 In Sect, 6.2.1, we began with the potential of a *perfect* dipole (Eq. 6.10), whereas in *fact* we are dealing with *physical* dipoles. Show, by the method of Sect. 4.2.3, that we nonetheless get the correct macroscopic field.